ELSEVIER Journal of Pure and Applied Algebra 115 (1997) 241–274

JOURNAL OF PURE AND APPLIED ALGEBRA

Generalized sketches as a framework for completeness theorems. Part III[1]

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Communicated by M. Barr; received 29 September 1993; revised 14 November 1994

1991 Math. Subj. Class.: Primary 03G30; Secondary 03C99, 03F99, 18C10

7. The sketch-specification of monoidal categories

The difficulty in sketch-specifying the concept of monoidal category is with arranging that the morphisms of sketches become in the doctrine the *not necessarily strict* monoidal functors. To achieve this effect, we modify the concept of monoidal category to what we call "anamonoidal category". The resulting concept can be directly treated by the approach of the present paper. Also, the ana-version has some other virtues pointed out below.

The idea is that all structure on a category should be determined *exactly* up to isomorphism; in our case, the unit object and the tensor products should be determined exactly up to isomorphism. This has two aspects. One is that, although (e.g.) the tensor of objects A and B is not unique, any two values of it are isomorphic. The other is that if C is isomorphic to (a value of) $A \otimes B$, then C itself is a possible value of $A \otimes B$. The right way of doing this involves "specifications", a new part of the structure.

I note that some of the ideas of this section appear, in a greater generality, in [27].

An *anamonoidal category* $\boldsymbol{C}$ is given by data (i) to (vii), subject to conditions (viii) to (xvi).

(i) An *underlying category*, also denoted by $\boldsymbol{C}$.

(ii) Abstract sets (classes) Un $[\boldsymbol{C}]$, Ten $[\boldsymbol{C}]$ of *unit specifications* and *tensor specifications*, respectively.

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[1] Research supported by NSERC Canada and FCAR Quebec. The abstract, the introduction and the references of Part I of this paper (this journal, Vol. 115 (1997) 49–79) apply to this concluding part as well. The numbering of the sections is consecutive throughout the three parts.

PII S0022-4049(96)00009-6

(iii) Each unit specification $u \in \mathrm{Un}[\boldsymbol{C}]$ picks out an object I_u, a *unit object*; and any two $u, v \in \mathrm{Un}[\boldsymbol{C}]$ pick out an isomorphism $\mathrm{i}_{u,v}: I_u \xrightarrow{\cong} I_v$.

(iv) Each tensor specification s picks out a triple $[s] = (A, B, C) = (\mathrm{A}_s, \mathrm{B}_s, \mathrm{C}_s)$ of three specific objects; we write $A \otimes_s B$ for C although A, B are also determined by s (that is, it is not the case that we have an operation $\otimes_s$ defined on all pairs (A, B)). It is possible that two *different* $s, t \in \mathrm{Ten}[\boldsymbol{C}]$ pick out the *same* triple $(\mathrm{A}_s, \mathrm{B}_s, \mathrm{C}_s) = (\mathrm{A}_t, \mathrm{B}_t, \mathrm{C}_t)$.

(v) Given two tensor specifications s, t, with corresponding objects $A, B, A \otimes_s B$ and $C, D, C \otimes_t D$, together with arrows $f: A \to C$, $g: B \to D$, we have the arrow $f \otimes_{s,t} g: A \otimes_s B \to C \otimes_t D$ determined by the pair (s, t).

(vi) $\lambda_{u,s}: I_u \otimes_s A \xrightarrow{\cong} A$, $\rho_{u,t}: A \otimes_t I_u \xrightarrow{\cong} A$ for any appropriate specifications $u \in \mathrm{Un}[\boldsymbol{C}]$, $s, t \in \mathrm{Ten}[\boldsymbol{C}]$ ("appropriate" means, for s, that $[s] = (I_u, A, I \otimes_s A)$).

(vii) With any four appropriate specifications s_0, s_1, s_2, s_3, giving rise to $A \otimes_{s_0} B$, $B \otimes_{s_1} C$, $A \otimes_{s_2} (B \otimes_{s_1} C)$, $(A \otimes_{s_0} B) \otimes_{s_3} C$, we have a corresponding *associativity isomorphism*

$$\alpha_{s_0, s_1, s_2, s_3}: A \otimes_{s_2} (B \otimes_{s_1} C) \xrightarrow{\cong} (A \otimes_{s_0} B) \otimes_{s_3} C$$

determined by (s_0, s_1, s_2, s_3).

(viii) (Existence of unit and of tensor) $\mathrm{Un}[\boldsymbol{C}]$ is inhabited: it is not empty; for any $A, B \in \mathrm{Ob}(\boldsymbol{C})$, there is at least one $s \in \mathrm{Ten}[\boldsymbol{C}]$ such that $\mathrm{A}_s = A$, $\mathrm{B}_s = B$.

(ix) $\mathrm{i}_{v,w} \circ \mathrm{i}_{u,v} = \mathrm{i}_{u,w}$ $(u, v, w \in \mathrm{Un}[\boldsymbol{C}])$.

(x) (Functoriality of the tensor) $1_A \otimes_{s,s} 1_B = 1_{A \otimes_s B}$ $(A, B \in \mathrm{Ob}(\boldsymbol{C})$, s giving rise to $A \otimes_s B)$; for any $a: A \to A'$, $a': A' \to A''$, $b: B \to B'$, $b': B' \to B''$ and specifications s, s', s'' giving rise to $A \otimes_s B$, $A' \otimes_{s'} B'$, $A'' \otimes_{s''} B''$, resp., we have $(a' \otimes_{s', s''} b') \circ (a \otimes_{s,s'} b) = (a' \circ a) \otimes_{s, s''} (b' \circ b)$.

(xi) (Naturality of λ and ρ)

$$\begin{tikzcd}
I_u \otimes_s A \arrow[r, "\lambda_{u,s}"] \arrow[d, "i_{u,v} \otimes_{s,t} f"'] \arrow[dr, phantom, "\circ"] & A \arrow[d, "f"] \\
I_v \otimes_t B \arrow[r, "\lambda_{v,t}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_p I_u \arrow[r, "\rho_{u,p}"] \arrow[d, "f \otimes_{p,q} i_{u,v}"'] \arrow[dr, phantom, "\circ"] & A \arrow[d, "f"] \\
B \otimes_q I_v \arrow[r, "\rho_{v,q}"'] & B
\end{tikzcd}$$

(xii) (Naturality of α)

$$\begin{tikzcd}
A \otimes_{s_2} (B \otimes_{s_1} C) \arrow[r, "\alpha_{s_0,s_1,s_2,s_3}"] \arrow[d, "a \otimes_{s_2,s_2} (b \otimes_{s_1,\dot{s}_1} c)"'] \arrow[dr, phantom, "\circ"] & (A \otimes_{s_0} B) \otimes_{s_3} C \arrow[d, "(a \otimes_{s_0,\dot{s}_0} b) \otimes_{s_3,\dot{s}_3} c"] \\
\dot{A} \otimes_{\dot{s}_2} (\dot{B} \otimes_{\dot{s}_1} \dot{C}) \arrow[r, "\alpha_{\dot{s}_0,\dot{s}_1,\dot{s}_2,\dot{s}_3}"'] & (\dot{A} \otimes_{\dot{s}_0} \dot{B}) \otimes_{\dot{s}_3} \dot{C}
\end{tikzcd}$$

(xiii) (Associativity coherence)

$$\begin{tikzcd}
A\otimes_2(B\otimes_1(C\otimes_0 D)) \arrow[rr, "1_A\otimes_{2,5}\alpha_{3,0,1,4}"] \arrow[d, "\alpha_{5,1,2,6}"'] & & A\otimes_5((B\otimes_3 C)\otimes_4 D) \arrow[dd, "\alpha_{9,4,5,10}"] \\
(A\otimes_5 B)\otimes_6(C\otimes_0 D) \arrow[d, "\alpha_{7,0,6,8}"'] \arrow[r, phantom, "\circ"] & {} & \\
((A\otimes_5 B)\otimes_7 C)\otimes_8 D & & (A\otimes_9(B\otimes_3 C))\otimes_{10} D \arrow[ll, "\alpha_{5,3,9,7}\otimes_{10,8}1_D"]
\end{tikzcd}$$

(xiv) (Triangular coherence)

$$\begin{tikzcd}
A\otimes_p(I_u\otimes_t B) \arrow[rr, "\alpha_{s,t,p,q}"] \arrow[dr, "1_A\otimes_{p,w}\lambda_{u,t}"'] & \circ & (A\otimes_s I_u)\otimes_q B \arrow[dl, "\rho_{u,s}\otimes_{q,w}1_B"] \\
& A\otimes_w B &
\end{tikzcd}$$

(xv) (*Unique transfer under isomorphism of unit specification*) For any $u\in \mathrm{Un}[\boldsymbol{C}]$ and isomorphism $\theta: I_u \xrightarrow{\cong} A$, there is a unique $v\in\mathrm{Un}[\boldsymbol{C}]$ such that $I_v = A$ and $\mathrm{i}_{u,v} = \theta$.

(xvi) (*Unique transfer under isomorphism of tensor specification*) Given $s\in\mathrm{Ten}[\boldsymbol{C}]$ giving rise to $A\otimes_s B$, and any isomorphism $i: A\otimes_s B \xrightarrow{\cong} C$, there is a unique $t\in\mathrm{Ten}[\boldsymbol{C}]$ such that $[t] = (A, B, C)$ and $i = 1_A\otimes_{s,t} 1_B$.

This says that if we have specified a tensor, and we have an isomorphism of the tensor to another object, we are free to re-specify the tensor as the new object, with marking the given isomorphism as the corresponding tensor of the identity arrows involved; moreover, the notion of tensor specification is "tight" in the sense that there is a *unique* new specification with the described properties.

This definition is rather complicated; what is its *raison d'etre*?

(A) We have the classical notion of monoidal category as described in [23, 8]. Given any monoidal category $(\boldsymbol{C}, I, \otimes, \alpha, \lambda, \rho)$, we can define an anamonoidal one, $\boldsymbol{C}_{(a)}$, as follows. We put $\mathrm{Ten}[\boldsymbol{C}_{(a)}]$ to consist of all entities of the form $s = (A, B, i: A\otimes B \xrightarrow{\cong} C) = (\mathrm{A}_s, \mathrm{B}_s, \mathrm{i}_s: \mathrm{A}_s\otimes\mathrm{B}_s \xrightarrow{\cong} \mathrm{C}_s)$; of course, $A\otimes_s B \underset{\mathrm{def}}{=} C$. Given $s, t\in\mathrm{Ten}[\boldsymbol{C}_{(a)}]$, and $a: \mathrm{A}_s\to\mathrm{A}_t$, $b: \mathrm{B}_s\to\mathrm{B}_t$, we define $a\otimes_{s,t} b$ by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{A}_s\otimes\mathrm{B}_s \arrow[r, "a\otimes b"] \arrow[d, "\mathrm{i}_s"'] \arrow[dr, phantom, "\circ"] & \mathrm{A}_t\otimes\mathrm{B}_t \arrow[d, "\mathrm{i}_t"] \\
\mathrm{C}_s \arrow[r, "a\otimes_{s,t} b"'] & \mathrm{C}_t
\end{tikzcd}$$

In particular, we have $s[A, B] \underset{\text{def}}{=} (A, B, 1_{A \otimes B}) \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, and we obtain that for any $s \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, $\mathrm{i}_s = (1_{\mathrm{A}_s} \otimes_{s, s[\mathrm{A}_s, \mathrm{B}_s]} 1_{\mathrm{B}_s})^{-1}$. The associativity isomorphisms are defined, very predictably, to make

$$\begin{tikzcd}
A\otimes_{s_2}(B\otimes_{s_1}C) \arrow[rr, "\alpha_{s_0,s_1,s_2,s_3}"] & & (A\otimes_{s_0}B)\otimes_{s_3}C \\
A\otimes(B\otimes_{s_1}C) \arrow[u, "\mathrm{i}_{s_2}"] & \circ & (A\otimes_{s_0}B)\otimes C \arrow[u, "\mathrm{i}_{s_3}"'] \\
A\otimes(B\otimes C) \arrow[rr, "\alpha_{A,B,C}"'] \arrow[u, "1_A\otimes \mathrm{i}_{s_1}"] & & (A\otimes B)\otimes C \arrow[u, "\mathrm{i}_{s_0}\otimes 1_c"']
\end{tikzcd}$$

commute. One easily sees that the unique transfer property for tensor specifications holds. I leave the definition of the rest of the structure of $\boldsymbol{C}_{(a)}$ and the verification of the axioms to the reader.

Conversely, given any anamonoidal category $\boldsymbol{C}$, we pick (axiom of choice), to any pair (A, B) of objects, a particular tensor specification $s = s_{A,B}$ with $\mathrm{A}_s = \mathrm{A}$, $\mathrm{B}_s = \mathrm{B}$, and define $A \otimes B = A \otimes_s B$. The rest of the data for a monoidal category $\boldsymbol{C}_{(s)}$ are now supplied easily. Note also that this does not use the unique transfer properties. Finally, one notes that, for $\boldsymbol{C}$ an anamonoidal category, $\boldsymbol{C}_{(s)(a)}$ is *isomorphic* to $\boldsymbol{C}$; for this we do have to use the unique transfer properties.

(B) The main argument in favor of the new concept is that it naturally gives the right notion of morphism of monoidal categories; this is the direct notion of a structure-preserving map. A morphism $F: \boldsymbol{C} \to \boldsymbol{D}$ of anamonoidal categories, an "anamonoidal functor", is the same as a functor between the underlying categories, together with maps $F: \mathrm{Un}[\boldsymbol{C}] \to \mathrm{Un}[\boldsymbol{D}]$, $F: \mathrm{Ten}[\boldsymbol{C}] \to \mathrm{Ten}[\boldsymbol{D}]$ which are compatible, in the straightforward sense, with the data. That is,

$$F(I_u) = I_{F(u)},$$

$$F(i_{u,v}) = i_{Fu,Fv},$$

$$F(A \otimes_s B) = (FA) \otimes_{Fs} FB,$$

$$F(a \otimes_{s,t} b) = (Fa) \otimes_{Fs,Ft} Fb,$$

$$F(\lambda_{u,s}) = \lambda_{F(u),F(s)}, \quad F(\rho_{u,s}) = \rho_{F(u),F(s)},$$

$$F(\alpha_{s_0,s_1,s_2,s_3}) = a_{Fs_0,Fs_1,Fs_2,Fs_3}.$$

Recall [8] the notion of a (not necessarily strict) monoidal functor $F: \boldsymbol{C} \to \boldsymbol{X}$. Consider the correspondence between the two types of monoidal category discussed above. If $\boldsymbol{C}, \boldsymbol{D}$ are monoidal, then we have a *bijection*

$$\hom(\boldsymbol{C}, \boldsymbol{D}) \to \hom(\boldsymbol{C}_{(a)}, \boldsymbol{D}_{(a)}) \tag{1}$$

(of course, the first hom-set is that of the monoidal functors, the second is that of the morphisms of anamonoidal categories): given $F \in \hom(\boldsymbol{C}, \boldsymbol{D})$, the corresponding $F_{(a)} \in \hom(\boldsymbol{C}_{(a)}, \boldsymbol{D}_{(a)})$ has its functor-part the same as F, and for $s = (A, B, i: A \otimes B \xrightarrow{\cong} C) \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, $F_{(a)}(s) = (FA, FB, j: FA \otimes FB \xrightarrow{\cong} FC)$ has $j = F(i) \circ \mathrm{i}_{A,B}$, the transition isomorphism $\mathrm{i}_{A,B}: FA \otimes FB \xrightarrow{\cong} F(A \otimes B)$ being given as part of the data for F. It is an instructive calculation to see that this mapping is indeed a bijection. In other words, the new definition of "anamonoidal category" *automatically* gives the right notion of morphism, which in the traditional framework is somewhat complicated in its appearance (and, e.g., is not given, presumably for that reason, in [23]).

(C) The concept of a morphism of parallel anamonoidal functors is also more natural than the usual formulation. If

$$\boldsymbol{C} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{D} \in \hom(\boldsymbol{C}, \boldsymbol{D}),$$

then $h: F \to G$ is a natural transformation between the functor-parts of F and G such that

(C.i) for any $u \in \mathrm{Un}[\boldsymbol{C}]$, $h_{\mathrm{I}_u} = \mathrm{i}_{Fu, Gu}$; and

(C.ii) for any $s \in \mathrm{Ten}[\boldsymbol{C}]$, $h_{A \otimes_s B}: F(A \otimes_s B) \to G(A \otimes_s B)$, i.e., $h_{A \otimes_s B}: FA \otimes_{Fs} FB \to GA \otimes_{Gs} GB$, is equal to $h_A \otimes_{Fs, Gs} h_B$.

Then, keeping in mind the usual notion for the monoidal case, we see that the mapping (1) is in fact an *isomorphism* of categories.

One feature of the non-strict notion of morphism of monoidal categories (in either the usual or the new sense) is the *(unique) transfer property for functors*: "any functor isomorphic to a given one is just as good as the given one". More precisely, given anamonoidal $\boldsymbol{C}$, $\boldsymbol{D}$ and $F \in \hom(\boldsymbol{C}, \boldsymbol{D})$, a functor $G: \boldsymbol{C} \to \boldsymbol{D}$ between the underlying categories, and $h: F \xrightarrow{\cong} G$, a natural isomorphism of functors, we can endow G with the structure of an anamonoidal functor $\boldsymbol{C} \to \boldsymbol{D}$, actually in a unique way that makes h into an arrow between parallel anamonoidal functors. Furthermore, if $\boldsymbol{C}$ is an anamonoidal category, $\boldsymbol{D}$ is a category, $F: \boldsymbol{C} \to \boldsymbol{D}$ is an equivalence-functor between categories, there is a way of endowing $\boldsymbol{D}$ with the structure of an anamonoidal category and F with the structure of an anamonoidal functor (actually equivalence).

(D) Perhaps the best thing for showing that the present approach to the definition of monoidal category is natural is to consider concrete examples. On p. 159 of [23], we find the description of the monoidal category $\langle \mathbf{Ab}, \otimes, \ldots \rangle$ of Abelian groups. A few lines above from this description, we find the parenthetical phrase "(any chosen)" in connection with Cartesian product, for the purposes of an example of monoidal category. The fact that a tensor product has to be *chosen* for each pair of Abelian groups in order to define $\langle \mathbf{Ab}, \otimes, \ldots \rangle$ is not mentioned, but it is clearly intended. The point of our approach is the avoidance of a choice. We define the anamonoidal category $\langle \mathbf{Ab}, \mathrm{Ten}[\mathbf{Ab}], \ldots \rangle$ by putting $\mathrm{Ten}[\mathbf{Ab}]$ to be the set of all entities $s = (A, B, f\colon A \times B \to C)$ such that f is a bilinear function *universal* among bilinear functions from $A \times B$ to Abelian groups; we put $\mathrm{A}_s = \mathrm{A}$, $\mathrm{B}_s = \mathrm{B}$ and $A \otimes_s B = C$. The rest of the definition should be clear. The definition satisfies the unique transfer properties, in particular the one for tensor specifications; this is a consequence of the fact that the universal bilinear arrow $f\colon A \times B \to C$ is determined up to a unique isomorphism. This fact is of course crucial to ensure that the new sense of **Ab** being an *ana*monoidal category is related to its being a monoidal category in the usual sense in the tight way described above.

On the basis of the above discussion, I adopt the position that anamonoidal categories are a good (or even better) substitute for monoidal categories, and proceed to exhibit their sketch-specification.

I remind the reader that, in Section 1, we have introduced the construction $\boldsymbol{G} \,\|\, \mathscr{G}$, which is an "indexed" version of the simpler one, $\boldsymbol{G} \,|\, \mathscr{G}$, and have even pointed out that the category $\boldsymbol{G} \,\|\, \mathscr{G}$ is "better"; it is a (finite, presheaf) topos, while $\boldsymbol{G} \,|\, \mathscr{G}$ is not (in general). Now, the construction $\boldsymbol{G} \,\|\, \mathscr{G}$ becomes essential.

For the notion of *(ana)monoidal sketch*, we start by taking $\mathbf{G} = \mathrm{cSk}$. Let Un be the c-sketch whose underlying graph has a single object, say 3, and no arrows; the specification sets I[Un], CT[Un] are empty. Let Ten be the c-sketch whose underlying graph has the three objects 0, 1 and 2, with no arrows; the specification sets I[Ten], CT[Ten] are empty. We will write $0 \otimes 1$ for 2, because of the use of Ten (to be clarified step by step).

Consider $\mathrm{cSk} \,\|\, \{\mathrm{Un}, \mathrm{Ten}\}$. An object S of $\mathrm{cSk} \,\|\, \{\mathrm{Un}, \mathrm{Ten}\}$ is a c-sketch $|S|$, together with abstract sets $\mathrm{Un}[S], \mathrm{Ten}[S]$, and assignments $u \mapsto \bar{u}, s \mapsto \bar{s}$ of a map $\bar{u}\colon \mathrm{Un} \to |S|$, $\bar{s}\colon \mathrm{Ten} \to |S|$ to each $u \in \mathrm{Un}[S]$, $s \in \mathrm{Ten}[S]$. Intuitively, an object of $\mathrm{cSk} \,\|\, \{\mathrm{Un}, \mathrm{Ten}\}$ is a c-sketch, together with specifications of instances of tensor-products and units, with the specifications having "individualizing" tags; thus, the very same map $\bar{s} = \bar{t}$ may correspond to two different $s, t \in \mathrm{Ten}[S]$.

Next, we perform a construction of the type $\boldsymbol{G} \,|\, \mathscr{G}$, with $\mathbf{G} = \mathrm{cSk} \,\|\, \{\mathrm{Un}, \mathrm{Ten}\}$. We will define objects $\widehat{\mathrm{U}}\mathrm{nArr}$, $\widehat{\mathrm{L}}\mathrm{eftUn}$, $\widehat{\mathrm{R}}\mathrm{ightUn}$, $\widehat{\mathrm{T}}\mathrm{enArr}$ and $\widehat{\mathrm{A}}\mathrm{ssoc}$, of $\boldsymbol{G}$. We will put

$$\mathrm{MonSk} \underset{\mathrm{def}}{=} \boldsymbol{G} \,|\, \{\widehat{\mathrm{U}}\mathrm{nArr}, \widehat{\mathrm{L}}\mathrm{eftUn}, \widehat{\mathrm{R}}\mathrm{ightUn}, \widehat{\mathrm{T}}\mathrm{enArr}, \widehat{\mathrm{A}}\mathrm{ssoc}\}, \tag{2}$$

the *category of monoidal sketches*. We will give sketch-entailments (arrows) $\mathscr{R}[\mathrm{Mon}]$ in MonSk so that the specification $(\mathrm{MonSk}, \mathscr{R}[\mathrm{Mon}])$ gives, as $\mathrm{MonSk} : \mathscr{R}[\mathrm{Mon}]$ (see Section 2), the category of anamonoidal categories (up to isomorphism). For the exposition, it is better to mix the descriptions of the specification-types (in the brackets in the definition of MonSk) and the axioms. I will number the axioms in the style (Ax*n*).

(Ax1) $\qquad \mathrm{Ex}[\mathrm{Un}] : \left|\ \boxed{}\ \right| \longrightarrow \left|\ \boxed{3}\ \right|$

$\mathrm{Un}[\,] = \{\mathrm{id}_{\mathrm{Un}}\}$

$\overline{\mathrm{id}}_{\mathrm{Un}} = \mathrm{id}_{\mathrm{Un}}$

What is the effect of Ex[Un]? A sketch S satisfies it iff there is a map $\varphi\colon \widehat{\mathrm{Ex}}[\mathrm{Un}] \to S$ (we continue the custom of using ^ for the codomain of a sketch-map), that is, iff there is $s \in \mathrm{Un}[S]$; note that φ necessarily maps 3 to the "value" $\bar{s}(3)$ of s in S. That is, Ex[Un] expresses the *existence* of (at least one) unit object.

$$\text{(Ax2)} \qquad \mathrm{Ex}[\mathrm{Ten}] : \left|\, 0 \quad 1 \,\right| \longrightarrow \begin{array}{c} \left|\, 0 \quad 1 \quad 0\otimes 1 \,\right| \\ \mathrm{Ten}[\ ] = \{\mathrm{id}_{\mathrm{Ten}}\} \\ \overline{\mathrm{id}}_{\mathrm{Ten}} = \mathrm{id}_{\mathrm{Ten}} \end{array}$$

From the specification-types in (2), let me start with $\widehat{\mathrm{Ten}}\mathrm{Arr}$; it has the following underlying c-sketch $|\widehat{\mathrm{Ten}}\mathrm{Arr}|$:

$$\begin{tikzcd} & 2 \arrow[dd, "{\langle 2,5\rangle}"] & \\ 0 \arrow[d, "{\langle 0,3\rangle}"'] & & 1 \arrow[d, "{\langle 1,4\rangle}"] \\ 3 & 5 & 4 \end{tikzcd}$$

with empty I[ ] and CT[ ]. In addition, $\widehat{\mathrm{Ten}}\mathrm{Arr}[\mathrm{Ten}] = \{\langle 0,1,2\rangle, \langle 3,4,5\rangle\}$; $|\langle 3,4,5\rangle| : \mathrm{Ten} \to |\mathrm{TenArr}|$ is the map for which $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto 5$; $|\langle 0,1,2\rangle|$ acts as the identity. We will write s, t for $\langle 0,1,2\rangle$ and $\langle 3,4,5\rangle$, resp., and

$$\begin{tikzcd} & 0\otimes_s 1 \arrow[dd, "a\otimes_{s,t} b"] & \\ 0 \arrow[d, "a"'] & & 1 \arrow[d, "b"] \\ \bar{0} & \bar{0}\otimes_t \bar{1} & \bar{1} \end{tikzcd}$$

for $\widehat{\mathrm{Ten}}\mathrm{Arr}$. Let TenArr^+ be the MonSk-sketch with underlying $\boldsymbol{G}$-sketch $\widehat{\mathrm{Ten}}\mathrm{Arr}$, and only additional specification $\mathrm{id} \in \widehat{\mathrm{Ten}}\mathrm{Arr}[\ ]$; and let $\ulcorner\mathrm{TenArr}\urcorner$ be

$$\begin{tikzcd} 0 \arrow[d, "a"'] & 0\otimes_s 1 & 1 \arrow[d, "b"] \\ \bar{0} & \bar{0}\otimes_t \bar{1} & \bar{1} \end{tikzcd}$$

that is, $\widehat{\mathrm{Ten}}\mathrm{Arr}$ without the middle arrow.

We adopt the axiom

$$\text{(Ax3)} \qquad \mathrm{Ex}[\mathrm{TenArr}] : \ulcorner\mathrm{TenArr}\urcorner \to \mathrm{TenArr}^+$$

(as usual, all arrows in specified sketch-axioms are inclusions as far as possible). Ex[TenArr] expresses the existence of the tensor-product of arrows.

The uniqueness of tensors of arrows is expressed by the codiagonal

$$\text{(Ax4)} \qquad \mathrm{Uni[TenArr]}: \mathrm{TenArr}^{+} \coprod_{\ulcorner \mathrm{TenArr} \urcorner} \mathrm{TenArr}^{+} \to \mathrm{TenArr}^{+}.$$

Based on the arrow UnArr: $\ulcorner$UnArr$\urcorner \to$ ÛnArr, in detail

$$\mathrm{UnArr}: \begin{array}{c}\boxed{\begin{array}{c}0\\ \\ 1\end{array}}\\ \mathrm{Un}[\,]=\{u,v\}\\ \bar{u}=0, \bar{v}=1\end{array} \longrightarrow \begin{array}{c}\boxed{\begin{array}{c}0\\ \downarrow i\\ 1\end{array}}\\ \mathrm{Un}[\,]=\{u,v\}\\ \bar{u}=0, \bar{v}=1\\ \mathrm{Iso}[\,]=\{i\}\end{array}$$

(here, for convenience, we used isomorphism specifications, adjoined to cSk as a first step; see Section 5; of course, its use here may be considered an abbreviation), the axioms

(Ax5) Ex[UnArr],

(Ax6) Uni[UnArr]

are formed similarly to the previous two.

Before describing Âssoc, note that Âssoc will have an underlying c-sketch |Âssoc|, and the latter has an underlying graph ||Âssoc||.

||Âssoc|| has six objects, $0, 1, 2, 0 \otimes 1, 1 \otimes 2, 0 \otimes (1 \otimes 2)$ and $(0 \otimes 1) \otimes 2$; it has four arrows: $a: 0 \otimes (1 \otimes 2) \to (0 \otimes 1) \otimes 2$, $a^{-1}: (0 \otimes 1) \otimes 2 \to 0 \otimes (1 \otimes 2)$, 1_a and $1_{a^{-1}}$. CT[|Âssoc|] and I[|Âssoc|] have specifications expressing that a, a^{-1} are inverses. Ten[Âssoc] has four elements, s_i for $i < 4$, to specify the four tensors in question; e.g., $\bar{s}_3$ maps $0 \mapsto 0$, $1 \mapsto 1 \otimes 2$, $2 \mapsto 0 \otimes (1 \otimes 2)$.

$\ulcorner$Assoc$\urcorner$ is the same as Âssoc but without a and a^{-1}. The axioms

(Ax7) Ex[Assoc],

(Ax8) Uni[Assoc]

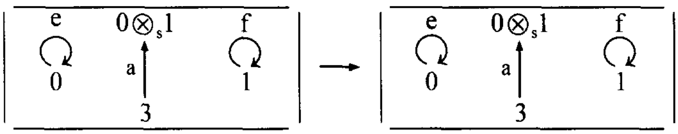
are then formed, based on $\ulcorner$Assoc$\urcorner \to$ Âssoc, as before. The axioms

(Ax9)–(Ax12) Ex[LeftUn], Un[LeftUn], Ex[RightUn], Un[RightUn]

are similar.

The axioms so far express clauses (i) to (viii) of the definition of anamonoidal category. Clauses (ix) to (xiv) are expressed by corresponding axioms (Ax13) to (Ax18) in a straightforward way, involving diagrams similar to those in the clauses themselves; I leave these to the reader. Finally, we have axioms corresponding to the transfer clauses (xv), (xvi).

(Ax19) IsoTransEx[Ten]:



Ten [ ] = {s}
Iso [ ] = {a}
I [ ] = {e,f}

t ∈ Ten [ ]:
$\bar{t}: 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 3$
$\sigma \in \widehat{\mathrm{Ten}}\mathrm{Arr}[\,]: \bar{\sigma}: \langle 0,3\rangle \mapsto e$,
$\bar{\sigma}: \langle 0,4\rangle \mapsto f, \bar{\sigma}: \langle 2,5\rangle \mapsto a$

(Ax20) $\mathrm{IsoTransUni[Ten]}: T \bigsqcup_S T \to T$,

with $S = \ulcorner\mathrm{IsoTransEx[Ten]}\urcorner$, $T = \widehat{\mathrm{I}}\mathrm{soTransEx[Ten]}$.

The preceding two axioms express the existence and uniqueness parts, respectively, in clause (xvi). Finally, we have axioms

(Ax21) IsoTransEx[Un],

(Ax22) IsoTransUni[Un]

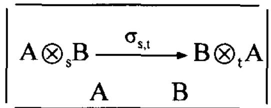
expressing (xv).

Any anamonoidal category gives rise to a MonSk–sketch, in the obvious way; the resulting sketch allows the unique reconstruction of the anamonoidal category; we may identify the two objects.

Inspection shows that we have:

an object of MonSk *is isomorphic to an anamonoidal category iff it satisfies all sketch-axioms of category, and all of the sketch-axioms of anamonoidal category* (Ax1) *to* (A × 22). *Moreover, under this identification, the morphisms of anamonoidal categories are bijectively mapped onto maps of sketches.*

Variants and extensions of the concepts of monoidal category can be treated in a similar way. Symmetric and braided monoidal categories will have additional specifications and axioms corresponding to the symmetry and the braiding, respectively. For example, the specification of the symmetry is by the sketch $\widehat{\mathrm{S}}\mathrm{ymm}$ =



Ten [ ] = {s,t}

symmetric monoidal categories will be specified in the sketch-category MonSk$|\{\hat{\text{S}}\text{ymm}\}$. Two of the axioms are Ex[Symm] and Uni[Symm], expressing the existence and the uniqueness of the symmetry; note that the symmetry depends on tensor specifications.

The treatment of closed monoidal categories is more understandable if it is based explicitly on [27]; I will not go into it here.

8. Kinds of completeness

For our purposes here, we will have to be specific about *size-restrictions*. In our previous definitions of categories, we restricted objects to be *small*; we had in mind (as usual) a certain fixed but arbitrary Grothendieck universe $\mathscr{U}_0$, and "small" meaning "belonging to $\mathscr{U}_0$". However, we want certain standard "large" categories such as the category of sets to be members of our doctrines. Therefore, we choose ("Grothendieck's axiom" [3]) another Grothendieck universe $\mathscr{U}_1$ with $\mathscr{U}_0 \in \mathscr{U}_1$, and take each of our sketch-categories and doctrines to have their objects restricted to $\mathscr{U}_1$ rather than $\mathscr{U}_0$; after all, $\mathscr{U}_0$ was arbitrary, the notion of "small" is relative. In other words, we decide that in the previous sections, "small" should be understood as $\mathscr{U}_1$-small; in particular, Set, the category of small sets as it appeared in the previous sections, should be understood as $\text{Set}_{\mathscr{U}_1}$, the category of $\mathscr{U}_1$-small sets. For the category of $\mathscr{U}_1$-small sets, and for that of $\mathscr{U}_0$-small sets, I will write SET and Set, respectively. Note that concepts such as "locally presentable", "accessible", etc., depend on the choice of the meaning of "small"; the said notions will be understood here with "small" meaning $\mathscr{U}_1$-small. Under these conventions, Set will be an object of the standard doctrines.

However, when a category is said to be lfp, we will insist that the class of isomorphism types of fp objects is (bijectively related to) a $\mathscr{U}_0$-small set. Likewise, in the definition of $(\boldsymbol{S}, \mathscr{R})$ being a finitary doctrine specification, we require that the data going into the description of $\boldsymbol{S}$ (the exponent-category $\boldsymbol{X}$ in the initial presheaf category $\text{SET}^{\boldsymbol{X}}$, the sets of specification names) and the rule-set $\mathscr{R}$ are $\mathscr{U}_0$-small.

In this section, the word "small" will be meant as "$\mathscr{U}_0$-small". With all other specific notations for categories (Graph, cSk, ...) appearing in the previous sections, we stick to the decision that they consist of the $\mathscr{U}_1$-small objects of the corresponding kind. Thus, the underlying graph of Set is an object of Graph; the category sketch associated with Set is in cSk, etc.

We could be very explicit about these things (in the style of [3]), and carry the universe-information with all our notations; however, this seemed burdensome to me.

These complications arise because in categorical representation theorems, categories of various sizes appear. For example, the embedding theorems for Abelian categories refer to arbitrary *small* Abelian categories, and the specific *large* categories of modules.

In Section 4, we gave the specification (cohSk, $\mathscr{R}$[Coh]) for coherent categories. We know (for references, see below) that Gödel's completeness theorem is translation-equivalent to the representation-theorem for (small) coherent categories; it is now my task to point out a *completeness theorem in terms of the sketch-based syntax* which is translation-equivalent to Gödel's completeness theorem.

There is an obvious guess we can make. Set, the category of small sets, is a member of the coherent doctrine cohSk: $\mathscr{R}$[Coh]; therefore, for any sketch-entailment σ in cohSk,

$$\vdash_{\mathscr{R}[\mathrm{Coh}]} \sigma \Rightarrow \mathrm{Set} \models \sigma.$$

If the reverse implication held, we would have a perfect completeness theorem! Unfortunately, it does not; not even for finite σ. Take the sketch-entailment

$$\left|\begin{array}{c} A \\ f\Big\downarrow \\ B \end{array}\right| \longrightarrow \left|\begin{array}{c} A \\ f\Big\downarrow \ \Big\uparrow g \\ B \end{array}\right|$$

$$\hat{\exists}[\ ] = \{\exists_f(1_A) = 1_B\} \qquad\qquad (g\,f = 1_B) \in CT[\ ]$$

expressing that "every surjective map has a section". This is valid in Set (axiom of choice), but it is not valid in all coherent categories; in particular, it is not deducible from $\mathscr{R}$[Coh]. The way out is to suitably restrict the class of entailments forming the range of σ. Recall exactness properties from Section 5. We have that

(1) *For any exactness property* σ *in* cohSk, *we have*

$$\vdash_{\mathscr{R}[\mathrm{Coh}]} \sigma \iff \mathrm{Set} \models \sigma.$$

In what follows, among others we will obtain a proof (1). In the last section, we will indicate why (1) is translation-equivalent to the Gödel completeness theorem (in the form saying that a sentence is deducible in the standard Hilbert-type formal system for first order logic iff it is valid).

In fact, we develop a framework for completeness theorems like (1), involving representation theorems. We will see, in a general context, that (1) follows from the General Completeness Theorem of Section 2, and the representation theorem for coherent categories:

(2) *For any small coherent category* $\boldsymbol{C}$, *there is a conservative* (*isomorphism reflecting*) *coherent functor* $\boldsymbol{C} \to \mathrm{Set}^I$ *into a suitable Cartesian power* Set^I *of* Set.

(For references concerning (2), see below.)

Let $\boldsymbol{S}$ be an 1fp category; of course, in practice, $\boldsymbol{S}$ will be a finitary sketch-category. Let $\mathscr{K}$ be a set of arrows between fp objects of $\boldsymbol{S}$. Let us call an arrow in $\boldsymbol{S}$ a $\mathscr{K}$-

statement, if it is a pushout of an arrow in $\mathscr{K}$. We introduce a notation to deal with the latter notion; with given

$$\begin{tikzcd} S \\ U \arrow[u, "\psi"] \arrow[r, "w"'] & V \end{tikzcd} \tag{3}$$

the (a chosen) pushout of the data is denoted as

$$\begin{tikzcd} S \arrow[r, "{\psi[w]}"] & S[w] \\ U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[ur, phantom, "\square"] & V \arrow[u, "\psi'"'] \end{tikzcd} \tag{3'}$$

A $\mathscr{K}$-statement is (up to isomorphism) one of the form $\psi[w]$ for some $w \in \mathscr{K}$ and arbitrary ψ. Note that every $\mathscr{K}$-statement is relatively finite (see Section 3).

Here is a class of examples for $\mathscr{K}$. Suppose that the sketch-category $\boldsymbol{S}$ is based on $\boldsymbol{G}$ (recall from Section 4 what this means). Consider the arrows of $\boldsymbol{S}$ of the form $\langle K \rangle \underset{\text{def}}{=} K \to K^+$, where K is a specification-type of $\boldsymbol{S}$ (over $\boldsymbol{G}$), and K^+ is obtained by adjoining a single element, the identity, to K in the specification set $K[\ ]$; that is, $K[K^+] = \{\mathrm{id}_K\}$, and the arrow is the inclusion. We may call the arrows described the *specification-arrows* of $\boldsymbol{S}$ (over $\boldsymbol{G}$). For example, when $\boldsymbol{S} = \mathrm{CcSk}$, the sketch-category for Cc categories, and $\boldsymbol{G} = \mathrm{Graph}$, then we have five specification-arrows, one for each of the specification types $\hat{\mathrm{I}}$, $\widehat{\mathrm{CT}}$, $\hat{\mathrm{t}}$, $\hat{\mathrm{P}}$ and $\widehat{\mathrm{Exp}}$; the specification-arrow corresponding to $\widehat{\mathrm{Exp}}$ is the inclusion

$$\begin{array}{ccc} \left|\begin{tikzcd} 0 & 2 \arrow[l, "\langle 2{,}0\rangle"'] \arrow[r, "\langle 2{,}1\rangle"] \arrow[d, "\langle 2{,}3\rangle"] & 1 \\ & 3 & \end{tikzcd}\right| & \longrightarrow & \left|\begin{tikzcd} 0 & 2 \arrow[l, "\langle 2{,}0\rangle"'] \arrow[r, "\langle 2{,}1\rangle"] \arrow[d, "\langle 2{,}3\rangle"] & 1 \\ & 3 & \end{tikzcd}\right| \\ \mathrm{P}[\ ] = \{\mathrm{i_p}\} & & \begin{array}{c} \mathrm{P}[\ ] = \{\mathrm{i_p}\} \\ \mathrm{Exp}[\ ] = \{\mathrm{id}\} \end{array} \end{array}$$

Let $\boldsymbol{K}$ be a set of specification-types of $\boldsymbol{S}$, and let $\mathscr{K} = \mathscr{K}[\boldsymbol{K}]$ be the set of all finite coproducts

$$\coprod_{i<n} \varphi_i : \coprod_{i<n} \ulcorner \varphi_i \urcorner \to \coprod_{i<n} \hat{\varphi}_i$$

of specification-arrows $\varphi_i : \ulcorner \varphi_i \urcorner \to \hat{\varphi}_i$. It is easy to see that when $\boldsymbol{K}$ is the set of all specification-types, the finite $\mathscr{K}[\boldsymbol{K}]$-statements are exactly the exactness properties of $\boldsymbol{S}$ (over $\boldsymbol{G}$), in the sense of Section 5. When $\mathscr{K} = \mathscr{K}[\boldsymbol{K}]$, we say "$\boldsymbol{K}$-statement" for $\mathscr{K}$-statement".

For a class $\boldsymbol{P}$ of sketches, the *$\boldsymbol{P}$-$\mathscr{K}$-statements* are those $\mathscr{K}$-statements whose domain is in $\boldsymbol{P}$. When $\mathscr{K} = \mathscr{K}[\boldsymbol{K}]$, we say "$\boldsymbol{P}$-$\boldsymbol{K}$-statement" for $\boldsymbol{P}$-$\mathscr{K}$-statement".

For data as in (3), let us write $S \vDash_{\psi} w$ to mean that there exists a commutative diagram

$$\begin{tikzcd} S & \\ U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[ur, phantom, "\circ"] & V \arrow[ul] \end{tikzcd}$$

In other words, $S \vDash w$ iff, for all $\psi : U \to S$, $S \vDash_{\psi} w$.

Let us say that the arrow $\varphi : S \to C$ is *$\mathscr{K}$-conservative*, if for all $(w : U \to V) \in \mathscr{K}$ and for all $\psi : U \to S$, $C \vDash_{\varphi \circ \psi} w$ implies that $S \vDash_{\psi} w$. The picture for this is as follows:

$$\begin{tikzcd} S \arrow[r, "\varphi"] & C \\ U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[ur, phantom, "\circ"] & V \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

For any $V \to C$ making the square commutative, there is a diagonal making the left triangle commutative. We recognize the notion of *purity*; see [1]; in other words, φ is $\mathscr{K}$-conservative is to say that φ is pure with respect to the arrows in $\mathscr{K}$, similarly to the terminology of [1]. We say "$\boldsymbol{K}$-conservative" for "$\mathscr{K}[\boldsymbol{K}]$-conservative".

In case $\mathscr{K} = \mathscr{K}[\boldsymbol{K}]$, this means that φ reflects the K-specifications for $K \in \boldsymbol{K}$; that is, for any specification-type $K \in \boldsymbol{K}$ and $\gamma : K \to S$, if $\varphi \circ \gamma \in K[C]$, then $\gamma \in K[S]$. If $\boldsymbol{K}$ consists of the single specification-type $\ulcorner \mathrm{Iso} \urcorner$ (see Section 5), then φ being $\boldsymbol{K}$-conservative means that φ reflects isomorphisms.

A family $\langle \varphi_i : S \to C_i \rangle_{i \in I}$ of arrows with a fixed domain S is *$\mathscr{K}$-conservative* if the induced arrow $S \to \prod_{i \in I} C_i$ is $\mathscr{K}$-conservative. This means that for all $(w : U \to V) \in \mathscr{K}$ and for all $\psi : U \to S$, if for all $i \in I$, $C_i \vDash_{\varphi_i \circ \psi} w$, then $S \vDash_{\psi} w$.

Let $\boldsymbol{C}$ and $\boldsymbol{A}$ be classes of objects of $\boldsymbol{S}$. We say that $\boldsymbol{C}$ is *$\mathscr{K}$-representative with respect to $\boldsymbol{A}$*, if for all $S \in \boldsymbol{A}$, the class of all arrows $S \to C$ with $C \in \boldsymbol{C}$ is $\mathscr{K}$-conservative. Spelled out, this means the following:

for any $(w : U \to V) \in \mathscr{K}$, $S \in \boldsymbol{A}$ and $\psi : U \to A$, if

for all $\varphi : S \to C$ with $C \in \boldsymbol{C}$, we have $C \vDash_{\varphi \circ \psi} w$, (4)

then $S \vDash_{\psi} w$.

We say "$\boldsymbol{K}$-representative" for "$\mathscr{K}[\boldsymbol{K}]$-representative".

Thus, when $\boldsymbol{S} = \mathrm{cohSk}$, $\boldsymbol{K} = \{\ulcorner \mathrm{Iso} \urcorner\}$, and $\boldsymbol{A}$ consists of the small objects of cohSk:$\mathscr{R}$[Coh], then $\boldsymbol{C} = \{\mathrm{Set}\}$ is $\mathscr{K}$-representative with respect to $\boldsymbol{A}$; this is the representation theorem (2) above. Note that in this case we could have chosen $\boldsymbol{K}$ as the set of all specification types, with equivalent results.

Let us make two observations. The first is that the condition in line (4) is equivalent to $\boldsymbol{C} \models \psi[w]$;

$$\forall C \in \boldsymbol{C},\ \forall \varphi : S \to C \quad (C \models_{\varphi \cdot \psi} w) \;\Leftrightarrow\; \boldsymbol{C} \models \psi[w] \tag{5}$$

(the data for this statement are as in (3)). To see this, assume the left-hand side of (5), and consider the diagram

$$\begin{tikzcd}
C & \\
S \arrow[u, "\varphi"] \arrow[r, "{\psi[w]}"] \arrow[ur, phantom, "\circ"] & S[w] \arrow[ul] \\
U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[ur, phantom, "\square"] & V \arrow[u]
\end{tikzcd}$$

in which we want the existence of the slanted arrow to make the triangle commute. But the assumption gives $V \to C$ making

$$\begin{tikzcd}
C & \\
S \arrow[u, "\varphi"] & \\
U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[ur, phantom, "\circ"] & V \arrow[uul]
\end{tikzcd}$$

commute; the required $S[w] \to C$ results from the universal property of the pushout. The converse direction of (5) is (even more) obvious. As a result of this observation, to say that $\boldsymbol{C}$ is $\mathscr{K}$-representative with respect to $\boldsymbol{A}$ is to say that, for data as in (3), and $S \in \boldsymbol{A}$,

$$\boldsymbol{C} \models \psi[w] \;\Leftrightarrow\; S \models_{\psi} w;$$

moreover, in this equivalence, the right-to-left implication is automatic; we have

(6) *$\boldsymbol{C}$ is $\mathscr{K}$-representative with respect to $\boldsymbol{A}$ iff for any $S \in \boldsymbol{A}$, $(w : U \to V) \in \mathscr{K}$ and* $\psi : U \to S$ (see (3)), *the implication*

$$\boldsymbol{C} \models \psi[w] \Rightarrow S \models_{\psi} w \tag{6'}$$

holds.

The second observation is that

(7) *if $\boldsymbol{C}$ is $\mathscr{K}$-representative for $\boldsymbol{A}$, then $\boldsymbol{C}$ is $\mathscr{K}^*$-representative for $\boldsymbol{A}$, where $\mathscr{K}^*$ is the class of all $\mathscr{K}$-statements.*

To see this, we show (6′) for $w \in \mathscr{K}^*$. We have a pushout as in the lower half of the following diagram, with $w_0 \in \mathscr{K}$:

$$
\begin{tikzcd}
S \arrow[r, "{\psi[w]}"] \arrow[dr, phantom, "\square"] & S[w] \\
U \arrow[u, "\psi"] \arrow[r, "w"'] \arrow[dr, phantom, "\square"] & V \arrow[u] \\
U_0 \arrow[u, "u"] \arrow[r, "w_0"'] & V_0 \arrow[u]
\end{tikzcd}
$$

But $\psi[w] = (\psi u)\,[w_0]$, and

$$C \models \psi[w] \;\Rightarrow\; C \models (\psi u)[w_0] \;\Rightarrow\; S \models_{\psi u} w_0 \;\Rightarrow\; S \models_{\psi} w,$$

where the second implication is by the assumed $\mathscr{K}$-representability, and the third by the universal property of the upper pushout.

Now, we introduce another parameter: a class $\mathscr{R}$ of finite entailments (arrows of $\boldsymbol{S}$); we will be interested in the doctrine $\boldsymbol{S}{:}\mathscr{R}$. Let $\boldsymbol{P}$ be a class of objects of $\boldsymbol{S}$. We say that the subclass $\boldsymbol{C}$ of $\boldsymbol{S}{:}\mathscr{R}$ *gives a complete semantics in $\boldsymbol{S}{:}\mathscr{R}$ for $\boldsymbol{P}$-$\mathscr{K}$-statements*, or that *$\boldsymbol{P}$-$\mathscr{K}$-completeness holds in $\boldsymbol{S}{:}\mathscr{R}$ with respect to $\boldsymbol{C}$*, if for any $\boldsymbol{P}$-$\mathscr{K}$-statement, deducibility from $\mathscr{R}$ is equivalent to validity in $\boldsymbol{C}$:

$$\vdash_{\mathscr{R}} \gamma \;\Leftrightarrow\; \boldsymbol{C} \models \gamma \tag{8}$$

for any $\boldsymbol{P}$-$\mathscr{K}$-statement γ.

As an example, (1) says that in cohSk:$\mathscr{R}$[Coh], $\boldsymbol{P}$-$\mathscr{K}$-completeness holds with respect to {Set}, with $\boldsymbol{P}$ the set of finite sketches, and $\mathscr{K} = \{\langle \mathrm{Iso} \rangle\}$.

A class of examples is obtained by choosing an infinite regular cardinal κ, and putting $\boldsymbol{P} = \boldsymbol{S}_\kappa$, the class of κ-presentable objects of $\boldsymbol{S}$. In the previous example, $\kappa = \aleph_0$.

We are going to state a result asserting the equivalence of completeness and representability in suitable senses. To have the result in sufficient generality for the applications, we bring in the concept of doctrinal hull; see 3.8. Let $\varphi_{\mathscr{U}} : S \to \ulcorner \mathscr{U} \urcorner$ be doctrinal hull of S, and let $\gamma : S \to T$. We note the equivalence

$$\models_{\mathscr{R}} \gamma \;\Leftrightarrow\; \ulcorner \mathscr{U} \urcorner \models_{\varphi_{\mathscr{U}}} \gamma. \tag{9}$$

Indeed, the left-to-right implication holds since $\ulcorner \mathscr{U} \urcorner \in \boldsymbol{S}{:}\mathscr{R}$. Conversely, assume $\ulcorner \mathscr{U} \urcorner \models_{\varphi_{\mathscr{U}}} \gamma$, $A \in \boldsymbol{S}{:}\mathscr{R}$, and $\varphi : S \to A$. By $\ulcorner \mathscr{U} \urcorner \models_{\varphi_{\mathscr{U}}} \gamma$, we have $\theta : T \to \ulcorner \mathscr{U} \urcorner$ such that $\theta \circ \gamma = \varphi_{\mathscr{U}}$. By 3.8. (ii), there is $\psi : \ulcorner \mathscr{U} \urcorner \to A$ such that $\psi \circ \varphi_{\mathscr{U}} = \varphi$. Thus, $(\psi \circ \theta) \circ \gamma = \varphi$, showing that $A \models_{\varphi} \gamma$.

Proposition 1. *Suppose $(\boldsymbol{S}, \mathscr{R})$ is a finitary doctrine specification, and $\mathscr{K}$ is a set of arrows between fp objects in $\boldsymbol{S}$. Let $\boldsymbol{P}$ be a class of objects in $\boldsymbol{S}$ such that for any $S \in \boldsymbol{P}$, any* *doctrinal hull of* S *(with respect to* $\mathscr{R}$*) is also in* $\boldsymbol{P}$. *Let* $\boldsymbol{C}$ *be a subclass of* $\boldsymbol{S}{:}\mathscr{R}$. *Then for* $\mathrm{h}_{\mathscr{R}}(\boldsymbol{P})$ *the class of doctrinal hulls of members of* $\boldsymbol{P}$,

$$\boldsymbol{C} \text{ is } \mathscr{K}\text{-representative with respect to } \mathrm{h}_{\mathscr{R}}(\boldsymbol{P}) \tag{10}$$

if and only if

$$\boldsymbol{P}\text{-}\mathscr{K}\text{-completeness holds in } \boldsymbol{S}{:}\mathscr{R} \text{ with respect to } \boldsymbol{C}. \tag{11}$$

Proof. Assume (11), to show (10). Because of (6), assume $S \in \mathrm{h}_{\mathscr{R}}(\boldsymbol{P})$, $w: U \to V$, $\psi: U \to S$ and $\boldsymbol{C} \models \psi[w]$, to show $S \models_{\psi} w$. By the right-to-left implication in (8), we have $\vdash_{\mathscr{R}} \psi[w]$ since, by assumption, $S \in \mathrm{h}_{\mathscr{R}}(\boldsymbol{P}) \subset \boldsymbol{P}$, and thus $\psi[w]$ is a $\boldsymbol{P}$-$\mathscr{K}$-statement. By soundness (see 3.2.; the $\vdash_{\mathscr{R}} s \Rightarrow \models_{\mathscr{R}} s$ direction of 3.3.), and $S \in \mathrm{h}_{\mathscr{R}}(\boldsymbol{P}) \subset \boldsymbol{S}:\mathscr{R}$, it follows that $S \models \psi[w]$, hence $S \models_{\psi} w$ (see (3′), with an arrow $\sigma: S[w] \to S$ witnessing $S \models_{\mathrm{id}_S} \psi[w]$, which gives $\sigma \circ \psi' : V \to S$ witnessing $S \models_{\psi} w$) as desired.

Assume (10), to show (11); let $\gamma: S \to T$ be a $\boldsymbol{P}$-$\mathscr{K}$-statement; we want to see that the equivalence (8) holds. The left-to-right implication follows from soundness since $\boldsymbol{C} \subset \boldsymbol{S}:\mathscr{R}$. Assume $\boldsymbol{C} \models \gamma$. Let $\varphi_{\mathscr{U}}: S \to \ulcorner \mathscr{U} \urcorner$ be a doctrinal hull of S with respect to $\mathscr{R}$. By 3.1. (iv), $\boldsymbol{C} \models \varphi_{\mathscr{U}}[\gamma]$. Applying (7), in particular (6′), in the form

$$\boldsymbol{C} \models \varphi_{\mathscr{U}}[\gamma] \;\Rightarrow\; \ulcorner \mathscr{U} \urcorner \models_{\varphi_{\mathscr{U}}} \gamma,$$

we obtain $\ulcorner \mathscr{U} \urcorner \models_{\varphi_{\mathscr{U}}} \gamma$. Applying (9), we get $\models_{\mathscr{R}} \gamma$. The General Completeness Theorem (3.3.) gives $\vdash_{\mathscr{R}} \gamma$ as desired. □

Recall from Section 2 the (small) cardinal $\lambda_{S,\mathscr{R}}$ associated with any finitary doctrine specification $(\boldsymbol{S}, \mathscr{R})$. When $\boldsymbol{P} = \boldsymbol{S}_{\kappa}$, $\kappa > \lambda_{S,\mathscr{R}}$, then the condition $\mathrm{h}_{\mathscr{R}}(\boldsymbol{P}) \subset \boldsymbol{P}$ of Proposition 1, is satisfied, by 3.8.(i). In fact, in this case $\mathrm{h}_{\mathscr{R}}(\boldsymbol{P}) = \boldsymbol{S}_{\kappa} \cap (\boldsymbol{S}{:}\mathscr{R}) \underset{\text{def}}{=} (\boldsymbol{S}{:}\mathscr{R})_{\kappa}$, since for every $S \in \boldsymbol{S}{:}\mathscr{R}$, S is its own (unique) doctrinal hull. Note that, by Section 2, $\boldsymbol{S}{:}\mathscr{R}$ is κ-accessible, and in fact

(12) *every object in* $\boldsymbol{S}{:}\mathscr{R}$ *is a* κ*-filtered colimit of objects in* $(\boldsymbol{S}{:}\mathscr{R})_{\kappa}$.

As a corollary, we arrive at the following simpler version of Proposition 1.

Corollary 2. *For* $(\boldsymbol{S}, \mathscr{R})$, $\mathscr{K}$ *and* $\boldsymbol{C}$ *as above, and* κ *a regular cardinal,* $\kappa > \lambda_{S,\mathscr{R}}$, *we have that* $\boldsymbol{C}$ *is* $\mathscr{K}$*-representative with respect to* $\boldsymbol{S}_{\kappa}$ *if and only if* $\boldsymbol{S}_{\kappa}$-$\mathscr{K}$*-completeness holds with respect to* $\boldsymbol{C}$.

The "only if" part of the corollary gives that (2) implies (1), in fact, more generally, (1) for σ an arbitrary small "exactness property", any small $\{\ulcorner\mathrm{Iso}\urcorner\}$-statement.

Above we said that (1) results from (2) and the General Completeness Theorem (GCT); here, via the proof of the proposition, we have brought in doctrinal hulls again, in addition to the GCT. Here is a proof of the "only if" part of the corollary without reference to doctrinal hulls (excepting the implicit ones in the proof of the GCT).

First, we observe that, under the conditions of the corollary,

$$\vdash_{\mathscr{R}} \gamma \;\Leftrightarrow\; (\boldsymbol{S}{:}\mathscr{R})_\kappa \vDash \gamma \tag{13}$$

for all $\boldsymbol{S}_\kappa$-$\mathscr{K}$-statements γ. The left-to-right implication is obvious. By the GCT, $\boldsymbol{S}:\mathscr{R} \vDash \gamma \Rightarrow \vdash_{\mathscr{R}} \gamma$. But by (12) any arrow from the domain of γ, a κ-presentable object in $\boldsymbol{S}$, into an object in $\boldsymbol{S}:\mathscr{R}$ factors through an object in $(\boldsymbol{S}:\mathscr{R})_\kappa$; hence, $(\boldsymbol{S}:\mathscr{R})_\kappa \vDash \gamma \Rightarrow \boldsymbol{S}:\mathscr{R} \vDash \gamma$. The right-to-left implication in (13) follows.

Assume that $\boldsymbol{C}$ is $\mathscr{K}$-representative with respect to $\boldsymbol{S}_\kappa$, γ is an $\boldsymbol{S}_\kappa$-$\mathscr{K}$-statement, $W \in (\boldsymbol{S}:\mathscr{R})_\kappa$, and $\theta : \mathrm{dom}(\gamma) \to W$. We have

$$\begin{array}{ccccc} \boldsymbol{C} \vDash \gamma & \Rightarrow & \boldsymbol{C} \vDash \theta[\gamma] & \Rightarrow & W \vDash_\theta \gamma \\ & \uparrow & & \uparrow & \\ & 3.1 & & (7) & \end{array}$$

This shows that

$$\boldsymbol{C} \vDash \gamma \;\Rightarrow\; (\boldsymbol{S}:\mathscr{R})_\kappa \vDash \gamma.$$

By (13), it follows that

$$\boldsymbol{C} \vDash \gamma \;\Rightarrow\; \vdash_{\mathscr{R}} \gamma,$$

which was to be shown.

The most frequent case of Corollary 2, in practice is obtained by taking $\kappa = \infty$, the least non-$\mathscr{U}_0$-small (strongly inaccessible) cardinal; the resulting assertion also follows from the conjunction of the cases of Corollary 2 for all $\kappa < \infty$.

Corollary 3. *Suppose $(\boldsymbol{S}, \mathscr{R})$ and $\mathscr{K}$ are as in Proposition* 1. *Then any class $\boldsymbol{C}$ objects in $\boldsymbol{S}:\mathscr{R}$ is $\mathscr{K}$-representative with respect to the small members of the doctrine $\boldsymbol{S}:\mathscr{R}$ if and only if $\boldsymbol{C}$ provides a complete semantics in $\boldsymbol{S}:\mathscr{R}$ with respect to all small $\mathscr{K}$-statements.*

I want to discuss the content of the last corollary in the case of first order logic. For this purpose, I bring in, besides the coherent doctrine (figuring in (1) and (2)), the doctrine of Boolean categories, BoolSk$:\mathscr{R}$[Bool] (Section 5), and that of finitary Grothendieck topologies, GrtopfinSk$:\mathscr{R}$[Grtopfin] (Section 6). Statement (2) above expresses that {Set} is $\{\langle \mathrm{Iso} \rangle\}$-representative with respect to the small members of the coherent doctrine. We have the further *representation theorems*

(14) {Set} *is* $\{\ulcorner \mathrm{Iso} \urcorner\}$*-representative with respect to the small members of the Boolean doctrine*

and

(15) {Set} *is* $\{\mathrm{Cov}_n \colon n \in \mathbb{N}\}$*-representative with respect to the small members of the doctrine of finitary Grothendieck topologies.*

For (15), note that a morphism of finitary Grothendieck topologies is $\{\mathrm{Cov}_n\colon n \in \mathbb{N}\}$-conservative iff every finite sieve (a family of maps with a common codomain) in the domain-category which is mapped into a cover by the morphism is a cover itself; briefly, "coverings are reflected". In (15), Set means the finitary Grothendieck site on the category Set consisting of the finite surjective sieves. Thus, (15) says that in a small finitary Grothendieck site $\mathscr{C}$, a finite sieve is a cover iff it becomes a cover under all site-maps from $\mathscr{C}$ to Set.

(15) is essentially identical to Deligne's theorem asserting that coherent toposes have enough points. Deligne's theorem appears as an Appendix to Exposé 6 of [3], and dates from 1963. It and its proof have nothing overtly to do with logic. In [24], a direct proof of (15) is given, without the detour through toposes. The proof in [24] is "purely representation theoretic", without overt reference to logic; in fact, it can be regarded as a "higher dimensional" variant of the standard proof of Stone's representation theorem for distributive lattices (in particular, for Boolean algebras).

Categorical logic, initiated by Lawvere, gradually arrived at the insight, mainly due to A. Joyal, that Deligne's theorem is *equivalent* to Gödel's completeness, more precisely, to the uncountable generalization of it given by Mal'cev in the 1930s. In [30], the proof of Deligne's theorem is given by applying Gödel completeness through a two-way *translation* process between logical operations and categorical operations; the precise spelling-out of the translation is in fact the main point of [30]. See also [16] for a reformulation of Deligne's proof, as well as a sketch of the argument of [30] (announced in [29]).

The representation theorem (2) for the coherent doctrine can be easily derived from (15); this is done in [24]. It also follows rather directly from Deligne's original theorem (via the "classifying topos" of the coherent category). (14) is a *special case* of (2), similarly to the fact that the Stone representation theorem for Boolean algebras is a special case of that for distributive lattices.

By Corollary 3, and as a consequence of (2), (14) and (15), respectively, we have:

(16) *For any small* $\{\ulcorner\mathrm{Iso}\urcorner\}$*-statement* σ *in* cohSk,

$$\vdash_{\mathscr{R}[\mathrm{Coh}]} \sigma \iff \mathrm{Set} \models \sigma.$$

(17) *For any small* $\{\mathrm{Cov}_n\colon n \in \mathbb{N}\}$*-statement* σ *in* GrtopfinSk,

$$\vdash_{\mathscr{R}[\mathrm{Grtopfin}]} \sigma \iff \mathrm{Set} \models \sigma.$$

(18) *For any small* $\{\ulcorner\mathrm{Iso}\urcorner\}$*-statement* σ *in* BoolSk,

$$\vdash_{\mathscr{R}[\mathrm{Bool}]} \sigma \iff \mathrm{Set} \models \sigma.$$

From the representation theorem (15) for Grothendieck topologies, we can also deduce that Set is representative with respect to the small members of the doctrines regSk : $\mathscr{R}$[Reg] (the doctrine of regular categories), exSk : $\mathscr{R}$[ex] (the doctrine of Barr-exact categories), dSk : $\mathscr{R}$[dis] (the doctrine of distributive categories), pretop : $\mathscr{R}$[pretop] (the doctrine of pretoposes). As consequences, we have the specific completeness theorems saying that in any of these doctrines, an (∞-)exactness property is deducible from the axioms of the doctrine iff it is valid in Set. The representation theorem with Set as the semantic object for flSk : $\mathscr{R}$[fl], the doctrine of categories with finite limits, is essentially trivial; the resulting specific completeness theorem is less so.

To state further completeness theorems, some remarks on exactness properties. In a doctrine of structured categories, a morphism is faithful in the usual sense iff it is $\{\mathrm{CT}\}$-conservative, equivalently, $\{\mathrm{I}, \mathrm{CT}\}$-conservative. For all doctrines of structured categories which contain in their specification the type of pullback as well as the axioms for pullback, and in which all operations are defined by universal properties (as in Section 4), $\{\ulcorner\mathrm{Iso}\urcorner\}$-conservativeness implies (hence, is equivalent to) full conservativeness, that is, $\boldsymbol{K}$-conservativeness for $\boldsymbol{K}$ the class of all specification-types. It requires more structure to have $\{\mathrm{CT}\}$-conservativeness (faithfulness) to be the same as full conservativeness; for pretoposes, this holds, but for coherent categories, it does not.

The conclusion in the sentence before the last is also valid for the doctrine of $\mathscr{F}$-categories (see Section 5); $\mathscr{F}$-categories form a doctrine of structured categories, but $\mathscr{F}$-quantification is not defined by a universal property.

The same, mutatis mutandis, can be said for the doctrine of S4-categories (Section 4); this is based on $\boldsymbol{G} = \mathrm{Graph}^{\rightarrow}$ rather than Graph.

In the next two paragraphs, "representative" means "$\{\ulcorner\mathrm{Iso}\urcorner\}$-representative with respect to small objects", the most common concept of representation in categorical logic.

The class of Grothendieck toposes are representative in the doctrine HeySk : $\mathscr{R}$[Hey] of Heyting categories, with respect to small objects and fully conservative maps. In fact, the subclass of presheaf-toposes $\mathrm{Set}^{\boldsymbol{C}}$ ($\boldsymbol{C}$ small category), and also the class of localic toposes $\mathrm{Sh}(L)$, with L a frame, is representative. These facts are essentially Kripke's completeness theorem [19] and Rasiowa's and Sikorski's completeness theorem for intuitionistic logic [32, 33], respectively. The class of presheaf toposes are representative in the doctrine biHeySk : $\mathscr{R}$[biHey] of biHeyting categories; see [31].

The S4-categories (see Section 5) of the form

$$\begin{array}{c}\mathrm{Set}^{|\boldsymbol{C}|}\\ \uparrow{\scriptstyle \cdot}\\ \mathrm{Set}^{\boldsymbol{C}}\end{array},$$

with $\boldsymbol{C}$ a small category, $|\boldsymbol{C}|$ its set of objects, and $(\ )^{\cdot}$ the canonical functor, are representative in the doctrine S4Sk : $\mathscr{R}[\square]$ of S4-categories. This is related to Kripke's completeness theorem for S4 modal predicate logic [18], but mathematically is somewhat stronger; for the exact form quoted here, see [31].

Several other completeness theorems in [31] are stated in the form of representation theorems for doctrines whose sketch-specifications should be easy to produce on the basis of the examples in this paper.

Keisler's completeness theorem [17] can be translated into saying that the $\mathscr{F}$-category Set with the $\kappa = \aleph_1$ interpretation is representative with respect to the class $\mathscr{B}$ of *countable* $\mathscr{F}$-categories (and with respect to the fully conservative maps); see also the next section. The corresponding sketch-based completeness theorem results by Corollary 2, with the choice $\kappa = \aleph_1$.

Next, we describe a representation theorem concerning Cartesian closed categories due to Cubric [5, 6], and the corresponding specific completeness theorem; this uses Proposition 1, rather than Corollaries 2 or 3.

In the case of the doctrine of Cartesian closed categories (Ccc's), we have the notion of the free Ccc over any graph, and, in fact, the *free Ccc over* any CcSk-sketch S as the Ccc $\mathscr{F}(S)$, with the sketch-map $\mathrm{i}_S : S \to \mathscr{F}(S)$, such that Ccc* $(\mathscr{F}(S), \boldsymbol{C}) \to$ CcSk $(S, \boldsymbol{C})$ defined by composition with i_S is an equivalence of categories. Here Ccc* $(\mathscr{F}(S), \boldsymbol{C})$ is the category of all Ccc-morphisms $\mathscr{F}(S) \to \boldsymbol{C}$, with all natural *isomorphisms* as arrows; and CcSk $(S, \boldsymbol{C})$ is the category whose objects are the sketch-maps $S \to \boldsymbol{C}$, and whose arrows are the natural *isomorphisms* between them as between maps from a graph to a category. $\mathscr{F}(S)$ is determined up to equivalence of categories.

It turns out that, up to equivalence of categories, the $\mathscr{F}(S)$ for a fixed S are the same as the doctrinal hulls of S. In other words,

(19) *if* $\mathrm{i}_{\mathscr{U}} : s \to \ulcorner \mathscr{U} \urcorner$ *is any doctrinal hull of* S, *then* $\mathrm{i}_{\mathscr{U}} : S \to \ulcorner \mathscr{U} \urcorner$ *has the universal property of* $\mathrm{i}_S : S \to \mathscr{F}(S)$.

This latter fact will be presented in [26] in a more general context.

Cubric's representation theorem concerns Ccc categories of the form $\mathscr{F}(G)$ for a *graph* G, that is, a CcSk-sketch with all specification-sets empty; let us call a Ccc of the form $\mathscr{F}(G)$ for a graph G a *free* Ccc. Cubric's theorem says that

(20) (Cubric) *Any free Ccc has a faithful Ccc functor to* Set.

Now, let us call a sketch S *free* if there are a graph G and an ∞-deduction $\psi : G \to S$. Consider the following class $\boldsymbol{P}$ of small CcSk-sketches. Since the composite of two composable ∞-deductions is again an ∞-deduction, therefore in case $\psi : G \to S$ is an ∞-deduction, any doctrinal hull of $\boldsymbol{S}$ is a doctrinal hull of G as well. Thus, the class $\boldsymbol{S}_{\text{free}}$ of free sketches is closed under taking doctrinal hulls. We conclude that 1. is applicable to obtain that

(21) $\mathscr{R}[\mathrm{Cc}]$ *is a complete axiomatization with respect to* $\boldsymbol{S}_{\text{free}}$-$\{\mathrm{CT}\}$-*statements in CcSk true in* Set

(a paraphrase of "$\boldsymbol{S}_{\text{free}}$-$\{\mathrm{CT}\}$-completeness holds in CcSk:$\mathscr{R}[\mathrm{Cc}]$ with respect to $\{\mathrm{Set}\}$").

The finite $\boldsymbol{S}_{\text{free}}$-$\{\mathrm{CT}\}$-statements are obtained by taking a finite graph G, extending G by applying the rules of Ccc's to obtain, say, $G \to S$, and, finally, arbitrarily equating two parallel arrows f and g in S, resulting in $G \to S \to S[f = g]$. The truth of the statement $G \to S[f = g]$ in a Ccc $\boldsymbol{C}$ is that "f and g are necessarily equal" in $\boldsymbol{C}$; that is, whenever G is interpreted in $\boldsymbol{C}$ via a map $\varphi : G \to \boldsymbol{C}$, inducing an essentially uniquely determined extension $\varphi' : S \to \boldsymbol{C}$, then $\varphi'(f) = \varphi'(g)$. (21) says that if $G \to S[f = g]$ is true in Set, then it is deducible from the axioms for Ccc's.

It is well-known that the restriction to the $\boldsymbol{S}_{\text{free}}$ cannot be removed from the Cubric completeness theorem; see [5, 6].

Let us turn to *compactness*. Consider a finitary sketch-category $\boldsymbol{S}$, and a class $\mathscr{K}$ of fp maps in $\boldsymbol{S}$ as before, but not any rule-set $\mathscr{R}$ as yet. A finite $\mathscr{K}$-statement α is a *finite* ($\mathscr{K}$-)*approximation* of the $\mathscr{K}$-statement φ if φ is a pushout of α; notice that any pushout of a $\mathscr{K}$-statement is a $\mathscr{K}$-statement. Let $\boldsymbol{P}$ and $\boldsymbol{C}$ classes of objects in $\boldsymbol{S}$. We say that ***P***-$\mathscr{K}$*-compactness holds with respect to* ***C*** if for any $\boldsymbol{P}$-$\mathscr{K}$-statement α, if α holds in $\boldsymbol{C}$, then there is a finite approximation of α which also holds in $\boldsymbol{C}$ (notice that the converse is automatic, and in fact that α is deducible from any approximation of it).

Proposition 4. *$\boldsymbol{P}$-$\mathscr{K}$-compactness holds with respect to $\boldsymbol{C}$ if and only if there is a set $\mathscr{R}$ of fp arrows in $\boldsymbol{S}$ such that $\boldsymbol{P}$-$\mathscr{K}$-completeness holds in $\boldsymbol{S}:\mathscr{R}$ with respect to $\boldsymbol{C}$.*

Proof. The "only if" part is obvious: take $\mathscr{R}$ to be the set of finite approximations of the $\boldsymbol{P}$-$\mathscr{K}$-statements that are true in $\boldsymbol{C}$.

For the converse, let $\mathscr{R}$ be a set of fp arrows. We show that if a $\mathscr{K}$-statement α is deducible from $\mathscr{R}$, then there is a finite approximation of α that is also deducible from $\mathscr{R}$; this will clearly suffice.

Assume n is an integer ≥ 0, and $\vec{r} = \langle r_i : R_i \to R_i' \rangle_{1 \leq i \leq n}$ is a sequence of fp arrows. An $\vec{r}$-*system* consists of data of the form

$$S_0 \xrightarrow{\varphi_1} S_1 \xrightarrow{\varphi_2} S_2 \xrightarrow{\varphi_3} S_3 \to \cdots \to S_{n-1} \xrightarrow{\varphi_n} S_n \tag{22.1}$$

and, for every $1 \leq i \leq n$, a pushout diagram

$$\begin{tikzcd} S_{i-1} \arrow[r, "\varphi_i"] & S_i \\ R_i \arrow[u, "\psi_i"] \arrow[r, "r_i"'] \arrow[ur, phantom, "\square"] & R_i' \arrow[u, "\psi_i'"'] \end{tikzcd} \tag{22.2}$$

The displayed $\vec{r}$-system is *finite* if each S_i is an fp object.

An arrow to the displayed $\vec{r}$-system from the $\vec{r}$-system

$$\hat{S}_0 \xrightarrow{\hat{\varphi}_1} \hat{S}_1 \xrightarrow{\hat{\varphi}_2} \hat{S}_2 \xrightarrow{\hat{\varphi}_3} \hat{S}_3 \to \cdots \to \hat{S}_{n-1} \xrightarrow{\hat{\varphi}_n} \hat{S}_n$$

$$\begin{tikzcd} \hat{S}_{i-1} \arrow[r, "\hat{\varphi}_i"] & \hat{S}_i \\ R_i \arrow[u, "\hat{\psi}_i"] \arrow[r, "r_i"'] \arrow[ur, phantom, "\square"] & R_i' \arrow[u, "\hat{\psi}_i'"'] \end{tikzcd}$$

is given by arrows $\sigma_i : \hat{S}_i \to S_i$ for $i < n$ such that

$$\begin{tikzcd}
S_{i-1} \arrow[rr, "\varphi_i"] & & S_i \\
\hat{S}_{i-1} \arrow[u, "\sigma_{i-1}"] \arrow[rr, "\hat{\varphi}_i"'] \arrow[urr, phantom, "\circ"] & & \hat{S}_i \arrow[u, "\sigma_i"']
\end{tikzcd}$$

and such that $\psi_i = \sigma_{i-1} \circ \hat{\psi}_i$, $\psi'_i = \sigma_i \circ \hat{\psi}'_i$ for all $1 \le i \le n$. Composition of arrows of $\vec{r}$-systems is obvious; thus, $\vec{r}$-systems form a category $\mathbf{S}_{\vec{r}}$. It is pretty clear that filtered colimits in $\mathbf{S}_{\vec{r}}$ exist, and they are computed componentwise. We prove by induction on the length n of $\vec{r}$ that

(23) *every $\vec{r}$-system is a directed colimit of finite $\vec{r}$-systems.*

For $n = 0$, the assertion holds since the object S is a directed colimit of fp objects. For the induction step, suppose the $\vec{r}$-system (22.1), (22.2) for $\vec{r}$ of length n, and assume (induction hypothesis) that the $\vec{r}' \underset{\text{def}}{=} \langle r_i : R_i \to R'_i \rangle_{1 \le i \le n-1}$-system obtained by deleting the data indexed by n is the colimit of a diagram of the finite $\vec{r}'$-systems

$$S_0^p \xrightarrow{\varphi_1^p} S_1^p \xrightarrow{\varphi_2^p} S_2^p \xrightarrow{\varphi_3^p} S_3^p \to \cdots \to S_{n-2}^p \xrightarrow{\varphi_{n-1}^p} S_{n-1}^p$$

$$\begin{tikzcd}
S^p_{i-1} \arrow[rr, "\varphi^p_i"] & & S^p_i \\
R_i \arrow[u, "\psi^p_i"] \arrow[rr, "r_i"'] \arrow[urr, phantom, "\square"] & & R'_i \arrow[u, "\psi'^p_i"']
\end{tikzcd}$$

indexed by p ranging over a directed poset P. Then, in particular, $S_{n-1} = \operatorname{colim}_p S^p_{n-1}$, with colimit coprojections $\iota^p_{n-1} : S^p_{n-1} \to S_{n-1}$, say. Since R_n is fp, there is $p_0 \in P$ such that $\psi_n : R_n \to S_{n-1}$ factors through $\iota^{p_0}_{n-1} : S^{p_0}_{n-1} \to S_{n-1}$ as $\psi_n = \iota^{p_0}_{n-1} \circ \psi^{p_0}_n$ for a suitable $\psi^{p_0}_n : R_n \to S^{p_0}_{n-1}$. Let $P_0 = \{p \in P : p_0 \le p\}$ be ordered as P is, a directed poset, and for $p \in P_0$,

$$\psi^p_n \underset{\text{def}}{=} \sigma^{p_0 \le p}_{n-1} \circ \psi^{p_0}_n : R_n \to S^p_{n-1},$$

where $\sigma^{p_0 \le p}_{n-1} : S^{p_0}_{n-1} \to S^p_{n-1}$ is the connecting map for the diagram of the S^p_{n-1}. Consider, for each $p \in P_0$, the pushout

$$\begin{tikzcd}
S^p_{n-1} \arrow[rr, "\varphi^p_n"] & & S^p_n \\
R_n \arrow[u, "\psi^p_n"] \arrow[rr, "r_n"'] \arrow[urr, phantom, "\square"] & & R'_n \arrow[u, "\psi'^p_n"']
\end{tikzcd} \tag{24}$$

defining S_n^p, φ_n^p and ψ'^p_n; note that S_n^p is fp. By the universal property of the pushout (24), we have $\iota_n^p : S_n^p \to S_n$ such that

$$\begin{tikzcd}
S_{n-1} \arrow[r, "\varphi_n"] \arrow[dr, phantom, "\circ"] & S_n \\
S^p_{n-1} \arrow[u, "\iota^p_{n-1}"] \arrow[r, "\varphi^p_n"] \arrow[dr, phantom, "\square"] & S^p_n \arrow[u, "\iota^p_n"'] \\
R_n \arrow[u, "\psi^p_n"] \arrow[r, "r_n"'] & R'_n \arrow[u, "\psi'^p_n"']
\end{tikzcd}$$

We also have similarly defined connecting maps $\sigma_n^{p \le q} : S_n^p \to S_n^q$. By (22.2) for $i = n$, it follows that $S_n = \operatorname{colim}_{p \in P_0} S_n^p$ with coprojections ι_n^p.

We have all the data needed to show that the given $\vec{r}$-system is indeed a directed colimit of finite $\vec{r}$-systems, which completes the proof of (23).

Now, suppose that the $\mathscr{K}$-statement $\alpha : S \to T$ is deducible from $\mathscr{R}$. Then, we have $n \ge 0$, a sequence $\vec{r} = \langle r_i : R_i \to R_i' \rangle_{1 \le i \le n}$ of arrows $r_i \in \mathscr{R}$, an $\vec{r}$-system (22.1), (22.2) with $S_0 = S$, and a diagram

$$\begin{tikzcd}
 & S_n \\
S \arrow[ur, "\varphi_{0n}"] \arrow[r, "\alpha"'] \arrow[ur, phantom, shift right=3, "\circ"] \arrow[dr, phantom, "\square"] & T \arrow[u, "\tau"'] \\
U \arrow[u, "u"] \arrow[r, "w"'] & V \arrow[u, "v"']
\end{tikzcd} \tag{25}$$

with $w \in \mathscr{K}$; here, $\varphi_{0n} = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$. Applying (23), let a P-indexed diagram of the finite $\vec{r}$-systems

$$S_0^p \xrightarrow{\varphi_1^p} S_1^p \xrightarrow{\varphi_2^p} S_2^p \xrightarrow{\varphi_3^p} S_3^p \to \cdots \to S_{n-1} \xrightarrow{\varphi_n^p} S_n^p$$

$$\begin{tikzcd}
S^p_{i-1} \arrow[r, "\varphi^p_i"] \arrow[dr, phantom, "\square"] & S^p_i \\
R_i \arrow[u, "\psi^p_i"] \arrow[r, "r_i"'] & R'_i \arrow[u, "\psi'^p_i"']
\end{tikzcd}$$

(one for each $p \in P$, P a directed poset) be such that its colimit is the given one, with coprojections $\iota_i^p : S_i^p \to S_i$. Since U and V are fp, there is $p \in P$ such that $u : U \to S$ factors as $u = \iota_0^p \circ u'$ for some $u' : U \to S_0^p$, and the composite $\tau v : V \to S_n$ factors as

$\tau v = \iota_n^p \circ \xi$ for a suitable $\xi : R_n \to S_n^p$ such that

$$\begin{tikzcd}
& S_n^p \\
S_0^p \arrow[ur, "\varphi^p_{0n}"] & \\
U \arrow[u, "u'"] \arrow[r, "w"'] \arrow[uur, phantom, "\circ"] & V \arrow[uu, "\xi"']
\end{tikzcd}$$

Let us consider the pushout

$$\begin{tikzcd}
S_0^p \arrow[r, "\theta'"] & T' \\
U \arrow[u, "u'"] \arrow[r, "w"'] \arrow[ur, phantom, "\square"] & V \arrow[u, "v'"']
\end{tikzcd}$$

This fits into the following two diagrams:

$$\begin{tikzcd}
& S_n^p \\
S_0^p \arrow[ur, "\varphi^p_{0n}"] \arrow[r, "\theta'"] & T' \arrow[u, "\eta"'] \\
U \arrow[u, "u'"] \arrow[r, "w"'] \arrow[ur, phantom, "\square"] & V \arrow[u, "v'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\alpha"] & T \\
S_0^p \arrow[u, "\iota_0^p"] \arrow[r, "\theta'"] \arrow[ur, phantom, "\square"] & T' \arrow[u, "\zeta"'] \\
U \arrow[u, "u'"] \arrow[r, "w"'] \arrow[ur, phantom, "\square"] & V \arrow[u, "v'"']
\end{tikzcd}
\tag{26}$$

($\varphi^p_{0n} = \varphi^p_n \circ \varphi^p_{n-1} \circ \cdots \circ \varphi^p_1$) such that, in addition, $\eta v' = \xi$ and $\zeta v' = v$ (the pushout character of the upper square of the second diagram follows from (25)). Then α' is an fp arrow, which is deducible from the first diagram in (26) (since φ^p_{0n} is directly deducible), and which is a finite $\mathscr{K}$-approximation of α by the second diagram in (26). This completes the proof of Proposition 4. $\square$

In the next section, we will show that in case the traditional concept of compactness is available, namely in the case of first order logic and its extensions with generalized quantifiers, our concept is equivalent to the traditional one.

Proposition 4 says that compactness is equivalent to *axiomatizability* by a set of *finite* axioms. The corresponding assertion in the traditional context is a well-known, and trivial, fact. Of course, the completeness theorems mentioned above as examples each have a corollary in the way of a compactness assertion.

In the traditional context, by *abstract completeness* we mean the recursive enumerability of validities. Abstract completeness is held up as the essential content of completeness, independent of a specific axiomatization. Of course, compactness is also such an "essential content", but less "basic" in some sense (certainly less finitary). As we noted in Section 3, when we are in a finite sketch-category, the set $\mathscr{D}[\mathscr{R}] = \{s : \vdash_{\mathscr{R}} s\}$ of formal consequences of an r.e. set $\mathscr{R}$ of rules is r.e. As it happens, in most cases in practice we have a *finite* set $\mathscr{R}$ of rules that axiomatizes the concept at hand. Of course, in all these cases, abstract completeness holds, where, formally, we say that, with respect to any given parameters $\boldsymbol{S}, \mathscr{K}, \boldsymbol{P}$ and $\boldsymbol{C}$ as before, *abstract completeness holds* if the set of finite $\boldsymbol{P}$-$\mathscr{K}$-statements valid in $\boldsymbol{C}$ form an r.e. set.

The value of the notions introduced in this section lies in their uniformity and conceptual simplicity. Now it is possible to ask, concerning categorically given logics, questions, the kind of which the logicians used to ask about logics defined by generalized quantifiers. For instance, consider the doctrine of finite limits and finite colimits (thus, this extends the doctrine of pretoposes by having arbitrary co-equalizers). Is the set of exactness properties with domain a free sketch (defined now in flcSk (see Section 4) as free sketches were in CcSk) that are true in Set recursively enumerable; finitely axiomatizable? There are a number of interesting questions of completeness concerning notions of monoidal categories. Our framework does not materially help in solving these problems, it only gives a convenient formulation to them.

9. Sketch-semantics versus Tarskian semantics

In this section, I explain how Tarski-type semantics can be related to sketch-semantics. This may contribute to a better understanding of sketch-semantics. We will see that compactness in the usual sense for a generalized quantifier with a specific semantics is equivalent to compactness in the sense of the previous section for appropriately chosen parameters. In the case when we have axiomatizability (as for instance we do for the case of the "there are uncountably many" quantifier), we will establish the existence of a recursive translation in both directions between deductions in the usual sense and in the sketch-based sense. We will stop short of spelling out an explicit translation of one kind of deduction into the other, although it would not be difficult to do it.

For specificity, let us consider (full, finitary, possibly multi-sorted) first order logic with (sorted) equality, with a new quantifier $\mathscr{F}x$ added. Thus, formulas are formed as in first order logic, together with a new formation-rule, allowing $\mathscr{F}x\varphi$ as a formula for any formula φ. For specificity, we consider $\mathbf{t}$ (true; 0-ary operator), $\wedge$, $\neg$ and $\exists$ as primitive, and the other first-order operators as abbreviations in the usual way. The notions of free and bounded variables, alphabetic variants, proper substitution and the like remain essentially the same as in ordinary first order logic; the $\mathscr{F}$-quantifier is handled syntactically just as $\exists x$ and $\forall x$ are.

A *semantics* of the $\mathscr{F}$ quantifier is specified by giving a class $\mathscr{S}$ of pairs (X, A) of sets, with A being a subset of X, subject to the condition of *invariance under isomorphism:*

whenever $f: X \xrightarrow{\cong} Y$ and $(X, A) \in \mathscr{S}$, then $(Y, f[A]) \in \mathscr{S}$. We write $\mathscr{S}[X] \underset{\text{def}}{=} \{A \subset X: (X, A) \in \mathscr{S}\}$.

We have the example, with any fixed cardinal κ, of the semantics $\mathscr{S}_\kappa$ consisting of those (X, A) in which the cardinality of A is less than κ. With $\kappa = \aleph_1$, this is the semantics we had in mind in Section 6 in connection with "$\mathscr{F}$-categories". A "dual" semantics $\mathscr{S}$ would have $(X, A) \in \mathscr{S}$ iff $\#(X - A) < \kappa$; still another dual would have $(X, A) \in \mathscr{S}$ iff $\# A \geq \kappa$. Both of these latter quantifiers are expressible using $\mathscr{F}$ in the first sense.

More "interesting" quantifiers can be introduced, by allowing the quantifier to bind more than one variable at a time, or even allowing quantification of variables ranging over subsets of the universe of the model, etc; see [4b]. The translations and conclusions given below have counterparts in the more involved situations as well.

Given a semantics $\mathscr{S}$ as described, we can evaluate formulas in any structure. From now on, we consider $\mathscr{S}$ fixed.

Let L be a language in the usual sense, that is, a collection of sorts, and sorted relation-symbols; for the sake of simplicity of exposition, we disallow operation-symbols. The notion of L-structure is the usual one. $\vec{x}, \vec{y}, \ldots$ always denote a proper variable sequence, that is, a finite sequence of *distinct*, sorted, variables. $\vec{x}$ is *of type* $\vec{X}$ if $\vec{x} = \langle x_i \rangle_{1 \leq i \leq n}$, $\vec{X} = \langle X_i \rangle_{1 \leq i \leq n}$ and $x_i : X_i$ (x_i is of sort X_i).

An entity $[\vec{x}: \varphi]$ (actually, simply an ordered pair $(\vec{x}, \varphi)$), where φ is a formula with all free variables in $\vec{x}$, is called an *(L-)formal set*, or *L-set*. Two formal sets $[\vec{x}: \varphi]$ and $[\vec{y}: \psi]$ are identified if $\vec{x}, \vec{y}$ are of the same type, and $\varphi[\vec{y}/\vec{x}]$, the result of properly (possibly renaming bound variables to avoid clashes) substituting $\vec{y}$ for $\vec{x}$, is an alphabetic variant of ψ (is obtained from ψ by properly renaming bound variables). The *type of* $[\vec{x}: \varphi]$ is the type of $\vec{x}$.

Given any L-structure M, $M[\vec{x}]$ denotes the Cartesian product $\prod_{i=1}^{n} M(X_i)$, where $\vec{x}$ is of type $\vec{X} = \langle X_i \rangle_{1 \leq i \leq n}$. The semantics $\mathscr{S}$ provides the definition of $M[\vec{x}: \varphi]$, or $M_{(\mathscr{S})}[\vec{x}: \varphi]$, the *($\mathscr{S}$-)meaning* of $[\vec{x}: \varphi]$ in M, a subset of $M[\vec{x}]$, by recursion on the structure of φ. The clauses with respect to the atomic formulas and the first-order logical operators are the usual ones. The specific clause related to $\mathscr{S}$ is this:

$$\vec{a} \in M[\vec{x}: \mathscr{F} y \varphi] \;\Leftrightarrow\; \{b: \vec{a}b \in M[\vec{x}y: \varphi]\} \in \mathscr{S}[M[\vec{x}]]$$

(here, it is assumed that y is not among $\vec{x}$; because of the possibility of renaming bound variables, this is not a real restriction; of course, $\vec{a}b$ denotes the concatenation $\vec{a}^\wedge \langle b \rangle$, and similarly for $\vec{x}y$). We say that M is an *$\mathscr{S}$-model* of the sentence σ, $M \models_{\mathscr{S}} \sigma$, or $M \models \sigma$ when $\mathscr{S}$ is understood, if $M[\emptyset: \sigma] = M[\emptyset: \mathbf{t}] = 1 = \mathbf{true}$. Models of T form a groupoid (category) $\mathrm{Mod}(T)$ (or $\mathrm{Mod}_{\mathscr{S}}(T)$), with morphisms the *iso*morphisms of L_T-structures; we ignore all morphisms but the isomorphisms.

We consider $\mathscr{F}$-sketches, objects in the sketch-category $\mathscr{F}\mathrm{Sk}$ defined in Section 6; of course $\mathscr{F}\mathrm{Sk}$ is independent of the semantics $\mathscr{S}$. However, $\mathscr{S}$ gives rise to a specific object $\mathrm{Set}_{(\mathscr{S})}$ in $\mathscr{F}\mathrm{Sk}$. As far as the specification-types for BoolSk (= cohSk) are concerned, $\mathrm{Set}_{\mathscr{S}}$ is given as the Boolean category Set. For the specification-type $\hat{\mathscr{F}}$ (see

Section 6), we set $(\Phi : \hat{\mathscr{F}} \to \mathrm{Set}) \in \hat{\mathscr{F}}[\mathrm{Set}_{(\mathscr{S})}]$ iff for all $y \in \Phi Y$,

$$\exists y' \in \Phi \mathrm{B}.\Phi \mathrm{b} y' = y. \iff \{x \in \Phi X \colon \exists x' \in \Phi \mathrm{A}.\Phi \mathrm{a} x' = x \,\&\, \Phi \mathrm{f} x = y.\} \in \mathscr{S}[\Phi \mathrm{X}]$$

(when Φa and Φb are inclusions of sets, which may be assumed without essential loss of generality, this becomes

$$y \in \Phi \mathrm{B} \iff \{x \in \Phi A \colon \Phi \mathrm{f} x = y\} \in \mathscr{S}[\Phi X]).$$

For any (small) $\mathscr{F}$-sketch S, the collection of sketch-maps $S \to \mathrm{Set}_{(\mathscr{S})}$, the *$\mathscr{S}$-models* of S, form a groupoid $\mathrm{Mod}(S)$ (or more explicitly, $\mathrm{Mod}_{(\mathscr{S})}(S)$), where a morphism $h : M \to N$ is, by definition, a (natural) *iso*morphism $h : M \xrightarrow{\cong} N$ between M and N as maps from the underlying graph $\|\mathscr{S}\|$ to the category Set (thus, we ignore all but the isomorphism mappings of models).

For L a language, an *L-fragment* F is a set of L-sets such that

$$[\vec{x} : \exists y \varphi] \in F \Rightarrow [\vec{x}y : \varphi] \in F,$$

$$[\vec{x} : \varphi \wedge \psi] \in F \Rightarrow [\vec{x} : \varphi] \in F \,\&\, [\vec{x} : \psi] \in F,$$

$$[\vec{x} : \neg \varphi] \in F \Rightarrow [\vec{x} : \varphi] \in F,$$

$$[xx' \colon x =_X x'] \in F \text{ for all sorts } X \ (x, x' \colon X),$$

$$[\emptyset : \mathbf{t}] \in F.$$

An *augmented theory* is a triple $T = (L, F, \Sigma) = (\mathrm{L}_T, \mathrm{F}_T, \Sigma_T)$, with L a language, F an L-fragment, Σ a set of L-sentences such that for $\sigma \in \Sigma$, $[\emptyset : \sigma] \in F$. (In contrast, a *simple theory* is a pair (L, Σ) of a language L and a set Σ of sentences; in what follows, until further notice, theories are augmented.) A sequence $\vec{X} = \langle X_i \rangle_{1 \le i \le n}$ of sorts *occurs* in T if either L_T has a relation-symbol R sorted as "$R \rightarrowtail X_1 \times \cdots \times X_n$", or if F_T contains a formal set $[\vec{x} : \varphi]$ of type $\vec{X}$. A *model* of T is a model of the corresponding simple theory (obtained by omitting the fragment); the notation $\mathrm{Mod}(T)$ is used accordingly.

With any augmented theory T, we associate an $\mathscr{F}$-sketch $S = \mathrm{S}[T]$; S consists of the following data (1.1) to (1.12).

(1.1) For every sort $X \in L$, an object X in S.

(1.2) For every $\vec{X} = \langle X_i \rangle_{1 \le i \le n}$ occurring in T, an object denoted $\vec{X}$, or $\prod_{i=1}^n X_i$, together with arrows $\pi_i : \prod_{i=1}^n X_i \to X_i$ $(1 \le i \le n)$, and the specification to the effect that $\langle \pi_i \rangle_i$ "is a product diagram". When $\vec{x}$ is of type $\vec{X}$, $[\vec{x}]$ means $\vec{X}$.

(1.3) For every relation-symbol "$R \rightarrowtail X_1 \times \cdots \times X_n$" in L, an object R and an arrow $R \overset{\mathrm{m}_R}{\rightarrowtail} \prod_{i=1}^n X_i$ in S, together with the specification that m_R "is a monomorphism".

(1.4) For every formal set $[\vec{x} : \varphi]$ in F, an object $[\vec{x} : \varphi]$ and a "monomorphism" $\mathrm{m}_{[\vec{x} : \varphi]} : [\vec{x} : \varphi] \to [\vec{x}]$.

(1.5) When $\Phi = [\vec{x} : \varphi] = [xx' \colon x =_X x']$, the specifications to the effect that $\mathrm{m}_\Phi : \Phi \rightarrowtail X \times X$ "is the diagonal".

(1.6) When $\Phi = [\vec{x}:\varphi]$ with φ the atomic formula $Ru_1 \cdots u_k$, with "$R \to U_1 \times \cdots \times U_k$", $\vec{x} = \langle x_i \rangle_{1 \le i \le n}$, $x_i : X_i$, $U_j = X_{i_j}$, $u_j = x_{i_j}$, a "pullback diagram"

$$\begin{tikzcd}
\Phi \arrow[r, tail, "\mathrm{m}_\Phi"] \arrow[d] & {[\vec{x}]} \arrow[d, "\pi"] \\
R \arrow[r, tail, "\mathrm{m}_R"'] & \vec{U}
\end{tikzcd}$$

together with the specification that π "is the morphism

$$\langle \pi_{i_j} \rangle_j : \prod_{i=1}^{n} X_i \to \prod_{j=1}^{k} U_j,$$

for the projections $\pi_{i_j} : \prod_{i=1}^{n} X_i \to X_{i_j}$".

(1.7) For every formal set of the form $[\vec{x}:\exists y\varphi]$ in F_T, a diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] & & {[\vec{x}y]} \arrow[dd, "f"] \\
{[\vec{x}y:\exists y\varphi]} \arrow[urr, "\mathrm{m}_{[\vec{x}y:\exists y\varphi]}"'] \arrow[d] & & \\
{[\vec{x}:\exists y\varphi]} \arrow[rr, "\mathrm{m}_{[\vec{x}:\exists y\varphi]}"'] & & {[\vec{x}]}
\end{tikzcd}$$

together with the specification that the rightmost vertical "is a product projection", and the data (in the form of the obvious map $\hat{\exists} \to |S|$ being an element of $\hat{\exists}[S]$) that the diagram is a $\exists$-specification.

(1.8) For every formal set of the form $[\vec{x}:\mathscr{F}y\varphi]$ in F_T, the diagram

$$\begin{tikzcd}
{[\vec{x}y:\varphi]} \arrow[r, tail, "\mathrm{m}_{[\vec{x}y:\varphi]}"] & {[\vec{x}y]} \arrow[d] \\
{[\vec{x}:\mathscr{F}y\varphi]} \arrow[r, tail, "\mathrm{m}_{[\vec{x}:\mathscr{F}y\varphi]}"'] & {[\vec{x}]}
\end{tikzcd}$$

understood as a map $\hat{\mathscr{F}} \to |S|$ in the obvious way is in $\hat{\mathscr{F}}[S]$.

(1.9), (1.10) and (1.11) Data explaining $[\vec{x}:\mathbf{t}]$, and ones relating $[\vec{x}:\varphi \wedge \psi]$ and $[\vec{x}:\neg\varphi]$ to $[\vec{x}:\varphi]$, $[\vec{x}:\psi]$ in the style of (1.7) and (1.8).

(1.12) For every $\tau \in \Sigma$, the specification that $\mathrm{m}_{[\emptyset:\tau]} : [\emptyset:\tau] \rightarrowtail [\emptyset:\mathbf{t}]$ "is an isomorphism".

Note that if T is finite (meaning that L_T, F_T, Σ_T are all finite), then $\mathrm{S}[T]$ is a finite sketch. Moreover, the map $T \mapsto \mathrm{S}[T]$, restricted to hf augmented theories T, is a recursive one.

We have a close connection between models of T and models of $\mathrm{S}[T]$. Every model M of T gives rise to a model $\hat{M}:\mathrm{S}[T]\to\mathrm{Set}_{(\mathscr{S})}$, for which

$\hat{M}X$ is MX for sorts X,

$\hat{M}(\prod_{i=1}^n X_i)$ is chosen to be the standard Cartesian product $\prod_{i=1}^n MX_i$ for the products $\prod_{i=1}^n X_i$ under (1.2),

For "$R\rightarrowtail\prod_{i=1}^n X_i$", $\hat{M}R$ is $MR(\subset\prod_{i=1}^n MX_i)$, and $\hat{M}(\mathrm{m}_R)$ is the inclusion $MR\to\prod_{i=1}^n MX_i$,

each $\hat{M}[\vec{x}:\varphi]$ is chosen to be equal to $M[\vec{x}:\varphi]$,

and the rest of the items for $\hat{M}$ also are chosen in the "standard" ways.

This map $M\mapsto\hat{M}$ extends to a functor

$$\mathrm{Mod}(T)\to\mathrm{Mod}(\mathrm{S}[T]). \tag{2}$$

To see this, we first characterize (iso)morphisms im $\mathrm{Mod}(\mathrm{S}[T])$.

(3) *Let $I,J\in\mathrm{Mod}(\mathrm{S}[T])$. Assume $\langle k_X: IX\xrightarrow{\cong}JX\rangle_X$ is a family of bijections indexed by the sorts in L_T such that, for every "$R\rightarrowtail X_1\times\cdots\times X_n$" in L, there is a (necessarily unique) k_R making*

$$\begin{tikzcd}
IR \arrow[r, "I\mathrm{m}_R"] \arrow[d, "k_R"'] \arrow[dr, phantom, "\circ"] & I(\prod_{i=1}^n X_i) \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "\circ"] & \prod_{i=1}^n IX_i \arrow[d, "\prod_{i=1}^n k_{X_i}"] \\
JR \arrow[r, "J\mathrm{m}_R"'] & J(\prod_{i=1}^n X_i) \arrow[r, phantom, "\cong"] & \prod_{i=1}^n JX_i
\end{tikzcd} \tag{4}$$

commute. Then there is a unique $h: I\xrightarrow{\cong}J$ for which $h_X=k_X$ for all $X\in\mathrm{Sort}[\mathrm{L}_T]$.

(Clearly, the condition on $\langle k_X\rangle_X$ is necessary for the existence of such h.)

Indeed, we define $h_X=k_X$, $h_{\vec{X}}=\prod_{i=1}^n k_{X^i}$, $h_R=k_R$ from (4). Next, we define the components $h_{[\vec{x}:\varphi]}: I[\vec{x}:\varphi]\xrightarrow{\cong}J[\vec{x}:\varphi]$ by recursion on the structure of $[\vec{x}:\varphi]$ so that

$$\begin{tikzcd}
I[\vec{x}:\varphi] \arrow[r, "I\mathrm{m}_{[\vec{x}\,\varphi]}"] \arrow[d, "h_{[\vec{x}\,\varphi]}"'] \arrow[dr, phantom, "\circ"] & I(\prod_{i=1}^n X_i) \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "\circ"] & \prod_{i=1}^n IX_i \arrow[d, "\prod_{i=1}^n k_{X_i}"] \\
J[\vec{x}:\varphi] \arrow[r, "J\mathrm{m}_{[\vec{x}\,\varphi]}"'] & J(\prod_{i=1}^n X_i) \arrow[r, phantom, "\cong"] & \prod_{i=1}^n JX_i
\end{tikzcd}$$

For this, we use the items (1.7) to (1.11), and the fact that the diagrams in those items are mapped by I and J into such diagrams in Set with known properties. Note especially that the recursion-step for the case of $[\vec{x}:\mathscr{F}y\varphi]$ uses the assumption of isomorphism invariance of $\mathscr{S}$. Finally, we need to define the components of h at certain auxiliary objects that come up in items (1.7) to (1.11), e.g., the P in (1.7); these are also uniquely determined under the naturality requirement.

To define the functor (2), we note that if $h: M \xrightarrow{\cong} N$, we can put $k_X = h_X$ and have that each diagram (4) commutes for suitable k_R since it is the same as

$$\begin{tikzcd} MR \arrow[r, "\mathrm{incl}"] \arrow[d, "h_R"'] \arrow[dr, phantom, "\circ"] & \prod_{i=1}^{n} MX_i \arrow[d, "\prod_{i=1}^{n} h_{X_i}"] \\ NR \arrow[r, "\mathrm{incl}"'] & \prod_{i=1}^{n} NX_i \end{tikzcd}$$

$\hat{h}: \hat{M} \to \hat{N}$ is given by (3) such that $\hat{h}_X = h_X$ for sorts X.

We have that the functor (2) is an equivalence of categories. In fact, we have that every model $I: T \to \mathrm{Set}_{(\mathscr{S})}$ gives rise to the L_T-structure $\tilde{I}$, for which $\tilde{I}X = IX$, and for "$R \rightarrowtail \prod_{i=1}^{n} X_i$" $\tilde{I}R$ is defined by the diagram

$$\begin{tikzcd} \tilde{I}R \arrow[r, tail, "\mathrm{incl}"] \arrow[d, "\cong"'] & \prod_{i=1}^{n} \tilde{I}X_i \arrow[d, "\cong"] \\ IR \arrow[r, tail, "Im_R"'] & I(\prod_{i=1}^{n} X_i) \end{tikzcd}$$

in Set where the right-hand vertical arrow is the canonical bijection between a standard Cartesian product and another product in Set. Moreover, $\tilde{I}$ is a model of T, and $\iota: (\tilde{I})\hat{} \cong I$ by a unique isomorphism ι whose components ι_X, X a sort, are identities; the unique existence of ι is ensured by (3).

Let now T be a simple theory $T = (\mathrm{L}_T, \Sigma_T)$, and σ and L_T-sentence. As usual, we write $T \models_{\mathscr{S}} \sigma$ for saying that σ is a consequence of T under the $\mathscr{S}$-semantics, that is, every $\mathscr{S}$-model M of T is an $\mathscr{S}$-model of σ.

Consider the augmented theory $T|\sigma \underset{\mathrm{def}}{=} (\mathrm{L}_T, F, \Sigma_T)$, where F is the least fragment for which $[\emptyset:\tau] \in F$ for all $\tau \in \Sigma$, and $[\emptyset:\sigma] \in F$; let $T \parallel \sigma \underset{\mathrm{def}}{=} (\mathrm{L}_T, F, \Sigma_T \cup \{\sigma\})$ with the same F. Consider the sketches $\mathrm{S}[T|\sigma]$, $\mathrm{S}[T \parallel \sigma]$. Note that the second is obtained from the first by adding a single isomorphism specification, specifying that $\mathrm{m}_{[\emptyset\cdot\sigma]}: [\emptyset:\sigma] \to [\emptyset:\mathbf{t}]$ is an isomorphism. Let $\alpha_{T,\sigma}$ be the inclusion $\mathrm{S}[T|\sigma] \to \mathrm{S}[T \parallel \sigma]$. Note that $\alpha_{T,\sigma}$ is an $\{\ulcorner\mathrm{Iso}\urcorner\}$-statement. I claim that

$$T \models_{\mathscr{S}} \sigma \iff \mathrm{Set}_{(\mathscr{S})} \models \alpha_{T,\sigma}. \tag{5}$$

First of all, we note that for $I \in \mathrm{Mod}(\mathrm{S}[T|\sigma])$ to have $\mathrm{Set}_{(\mathscr{S})} \models_I \alpha_{T,\sigma}$ is the same as to have that $I(\mathrm{m}_{[\emptyset:\sigma]})$ is an isomorphism. It is clear that this property is invariant under isomorphism in $\mathrm{Mod}(\mathrm{S}(T|\sigma])$; if $I \cong J$, and $\mathrm{Set}_{(\mathscr{S})} \models_I \alpha_{T,\sigma}$, then $\mathrm{Set}_{(\mathscr{S})} \models_J \alpha_{T,\sigma}$. Now, assume that $T \models_{\mathscr{S}} \sigma$, and $I \in \mathrm{Mod}(\mathrm{S}[T|\sigma])$. Let $M = \tilde{I}$. Since $M \models T$, we have that $M \models \sigma$. Consider $J = \hat{M}$. By the definition of $\hat{M}$, and $M \models \sigma$, we have that $J(\mathrm{m}_{[\emptyset:\sigma]})$ is an isomorphism, that is, $\mathrm{Set}_{(\mathscr{S})} \models_J \alpha_{T,\sigma}$. But, $J \cong I$, hence also $\mathrm{Set}_{(\mathscr{S})} \models_I \alpha_{T,\sigma}$ as was to be shown. The converse is similar and simpler.

Let us pause to consider the equivalence (5).

First of all, we have a version for pure first order logic, without the $\mathscr{F}$-quantifier; to obtain this, in the above, ignore all references to $\mathscr{F}$ and $\mathscr{S}$. We have

$$T \models \sigma \iff \mathrm{Set} \models \alpha_{T,\sigma}, \tag{6}$$

for any first-order theory T, sentence σ in the language of T, and the corresponding BoolSk-sketch entailment $\alpha_{T,\sigma}$. Now, let us apply *both* the Gödel–Mal'cev completeness theorem and the sketch-based completeness theorem 8.(18) for first-order logic. First, we apply Gödel's to conclude that

$$T \vdash \sigma \iff T \models \sigma, \tag{7}$$

where "$T \vdash \sigma$" means that "σ is deducible in T" in the standard formal system. Second, 8.(18) says that

$$\vdash_{\mathscr{R}[\mathrm{Bool}]} \alpha_{T,\sigma} \iff \mathrm{Set} \models \alpha_{T,\sigma}. \tag{8}$$

We conclude that

$$T \vdash \sigma \iff \vdash_{\mathscr{R}[\mathrm{Bool}]} \alpha_{T,\sigma}. \tag{9}$$

In other words, there is a deduction d of σ in T in the ordinary sense iff there is a sketch-deduction d' of $\alpha_{T,\sigma}$ from the rules $\mathscr{R}[\mathrm{Bool}]$. Given that the construction of $\alpha_{T,\sigma}$ from T and σ is effective, and both notions of deduction are effective, we find that there are recursive functions assigning a d' to any d, and another one, assigning a d to any d'; in other words,

(10) *there are recursive functions of the variables T and σ*, a finite theory and a sentence, respectively, in classical first order logic, *assigning a sketch-deduction of $\alpha_{T,\sigma}$ from $\mathscr{R}[\mathrm{Bool}]$ to any ordinary deduction of σ from T, and vice versa.*

It is possible to describe such recursive functions quite explicitly, by extending the translation process started above to deductions.

The above translation of symbolic logic into sketch-logic allows us to infer compactness or abstract completeness for symbolic logic from the same-named property (understood in the sense of the last section) for sketch-logic.

A *subtheory* of the (simple) theory T is any (simple) theory T' such that $\mathrm{L}_{T'} \subset \mathrm{L}_T$ and $\Sigma_{T'} \subset \Sigma_T$. Let us write $T' \leq T$ if this holds. For any theory T, the finite subtheories of T, ordered under $\leq$, form a directed poset. Let T and the L_T-sentence

σ be fixed, and let us write $T' \leq_\sigma T$ if $T' \leq T$, T' is finite, and σ is an $\mathrm{L}_{T'}$-sentence. For any $T' \leq_\sigma T$, we have the inclusion maps $\iota_{T'}: \mathrm{S}[T'|\sigma] \to \mathrm{S}[T|\sigma]$, $\tilde{\iota}_{T'}: \mathrm{S}[T' \parallel \sigma] \to \mathrm{S}[T \parallel \sigma]$. It is easy to see that

(11) $\mathrm{S}[T|\sigma] = \operatorname*{colim}_{T' \leq_\sigma T} \mathrm{S}[T'|\sigma]$,

with colimit coprojections the $\iota_{T'}$, and similarly for $\mathrm{S}[T \parallel \sigma]$. Now, let κ be a cardinal, possibly ∞, and let us consider the property of *κ-compactness* for the $\mathscr{S}$-semantics: this holds if for any theory T of cardinality $< \kappa$ in first-order logic extended with $\mathscr{F}$, and any L_T-sentence σ, $T \models_{\mathscr{S}} \sigma$ iff for some $T' \leq_\sigma T$ (in particular, T' is *finite*), $T' \models_{\mathscr{S}} \sigma$. We have that, for κ any uncountable regular cardinal,

(12) *κ-compactness for $\mathscr{S}$-semantics is implied by $\boldsymbol{S}_\kappa$-$\{\ulcorner\mathrm{Iso}\urcorner\}$-compactness for $\{\mathrm{Set}_{(\mathscr{S})}\}$ in $\mathscr{F}$*Sk.

Indeed, assume $\boldsymbol{S}_\kappa$-$\{\ulcorner\mathrm{Iso}\urcorner\}$-compactness for $\{\mathrm{Set}_{(\mathscr{S})}\}$ in $\mathscr{F}$Sk, and let T be a theory of cardinality $< \kappa$ such that $T \models_{\mathscr{S}} \sigma$. Then, by (5), $\mathrm{Set}_{(\mathscr{S})} \models \alpha_{T,\sigma}$. $\alpha_{T,\sigma}$ is an $\boldsymbol{S}_\kappa$-$\{\ulcorner\mathrm{Iso}\urcorner\}$-statement. Hence, there is a finite $\{\ulcorner\mathrm{Iso}\urcorner\}$-approximation $\beta: U \to V$ of $\alpha_{T,\sigma}$,

$$\begin{tikzcd}
\mathrm{S}[T|\sigma] \arrow[r, "\alpha_{T,\sigma}"] & \mathrm{S}[T \parallel \sigma] \\
U \arrow[u, "u"] \arrow[r, "\beta"'] \arrow[ur, phantom, "\square"] & V \arrow[u, "v"']
\end{tikzcd}$$

such that $\mathrm{Set}_{(\mathscr{S})} \models \beta$. By (11), and since U is finite, there is $T' \leq_\sigma T$ such that we have the diagram

$$\begin{tikzcd}
\mathrm{S}[T|\sigma] \arrow[r, "\alpha_{T,\sigma}"] & \mathrm{S}[T \parallel \sigma] & \\
\mathrm{S}[T'|\sigma] \arrow[u, "\iota_{T'}"] \arrow[r, "\alpha_{T',\sigma}"] & \mathrm{S}[T' \parallel \sigma] \arrow[u, "\tilde{\iota}_{T'}"'] & \\
U \arrow[u, "u'"] \arrow[r, "\beta"'] \arrow[ur, phantom, "3\square"] & V \arrow[u, "v'"'] \arrow[r, phantom, "2\square"] & {} \\
\ulcorner\mathrm{Iso}\urcorner \arrow[u] \arrow[r, "\langle \ulcorner\mathrm{Iso}\urcorner \rangle"] \arrow[ur, phantom, "1\square"] & \ulcorner\mathrm{Iso}\urcorner^{+} \arrow[u] &
\end{tikzcd}$$

in which $\iota_{T'} \circ u' = u$ and $\tilde{\iota}_{T'} \circ v' = v$; the fact of the pushout marked 3 follows from those marked 1 and 2. We conclude that $\mathrm{Set}_{(\mathscr{S})} \models \alpha_{T',\sigma}$, which, by (5) again, means $T' \models_{\mathscr{S}} \sigma$ as desired.

We also have, almost immediately, from (5) that:

(13) *Abstract completeness for first-order logic with the $\mathscr{S}$-quantifier is implied by abstract completeness for* $\boldsymbol{S} = \mathscr{F}\mathrm{Sk}$, $\mathscr{K} = \mathscr{K}[\{\ulcorner\mathrm{Iso}\urcorner\}]$, $\boldsymbol{P} = \boldsymbol{S}_{\aleph_0}$, *and* $\boldsymbol{C} = \{\mathrm{Set}_{(\mathscr{S})}\}$.

We also have the converses of (12) and (13) (obtained by replacing "is implied by" with "implies"). To prove these statements, we need to develop a translation of sketch-semantics into Tarskian semantics.

To any $\mathscr{F}\mathrm{Sk}$-sketch S, we associate a simple theory $\mathrm{T}[S] = (\mathrm{L}_S, \Sigma_S)$ such that we have an *isomorphism*

$$\mathrm{Mod}_{(\mathscr{S})}(S) \cong \mathrm{Mod}_{(\mathscr{S})}(T[S]) \tag{14}$$

of groupoids. The language L_S of $\mathrm{T}[S]$ is, by definition, the underlying graph of S; that is, the sorts of L_S are the objects of S, L_S has a unary operation symbol "$f: A \to B$" for each arrow $f: A \to B$ of S, and L_S has no other symbols. The axioms of $\mathrm{T}[S]$ correspond to the specifications in S; each specification of S is "expressed" by an axiom. In $\mathscr{F}\mathrm{Sk}$, we have the specification-type $\hat{\mathrm{t}}$, $\hat{\mathrm{P}}\mathrm{b}$, $\ulcorner\mathrm{Iso}\urcorner$, $\ulcorner\mathrm{Mon}\urcorner$, $\hat{\exists}$, $\hat{\imath}$, $\hat{\jmath}\mathrm{oin}$ and $\hat{\mathscr{F}}$. We will give the axioms corresponding to the $\hat{\mathrm{P}}\mathrm{b}$-and the $\hat{\mathscr{F}}$-specifications. For the rest, the reader may consult [30, pp. 90–92], where axioms are used for a similar task of "expressing". Recall that $\hat{\mathrm{P}}\mathrm{b}$ is

$$\begin{tikzcd}
0 \arrow[r, "f"] & 2 \\
3 \arrow[u, "p_1"] \arrow[r, "p_2"'] \arrow[ur, "h"'] & 1 \arrow[u, "g"']
\end{tikzcd}$$

$\mathrm{CT}[\;] = \{\mathrm{fp}_1 = \mathrm{h}, \mathrm{gp}_2 = \mathrm{h}\}$

For each $\varphi \in \hat{\mathrm{P}}\mathrm{b}\,[S]$, we adjoin the following axiom to Σ_S (here and below, for easier reading, we write $\bar{a}$ for $\varphi(a)$ for all arguments a of φ):

$$\forall z:\bar{3}.\bar{\mathrm{f}}\bar{\mathrm{p}}_1 z = \bar{\mathrm{h}}z \& \bar{\mathrm{g}}\bar{\mathrm{p}}_2 z = \bar{\mathrm{h}}z. \;\&\; \forall x:\bar{0}.\forall y:\bar{1}.(\bar{\mathrm{f}}x = \bar{\mathrm{g}}y \to \exists! z:\bar{3}.\bar{\mathrm{p}}_1 z = x \& \bar{\mathrm{p}}_2 z = y.).$$

Recall that $\hat{\mathscr{F}}$ is

$$\begin{tikzcd}
A \arrow[r, tail, "a"] & X \arrow[d, "f"] \\
B \arrow[r, tail, "b"'] & Y
\end{tikzcd}$$

$\mathrm{mon}[\;] = \{\mathrm{a}, \mathrm{b}\}$

Given $\varphi \in \hat{\mathscr{F}}[S]$, $\tilde{\mathrm{A}}x$ abbreviates the formula $\exists x' \in \bar{\mathrm{A}}.\bar{\mathrm{a}}x' = x$, and similarly for $\tilde{\mathrm{B}}y$. For each $\varphi \in \hat{\mathscr{F}}[S]$, we adjoin the following axiom to Σ_S:

$$\forall y:\bar{Y}(\tilde{\mathrm{B}}y \leftrightarrow \mathscr{F}x:\bar{\mathrm{X}}(\tilde{\mathrm{A}}x \& \bar{\mathrm{f}}x = y)).$$

Notice that both $\mathrm{Mod}_{(\mathscr{S})}(S)$ and $\mathrm{Mod}_{(\mathscr{S})}(\mathrm{T}[S])$ are subcategories of $\mathrm{Hom}(\|S\|, \mathrm{Set}) = \mathrm{Str}(\mathrm{L}_S)$; the graph-maps $\|S\| \to \mathrm{Set}$ are the same things as the L_S-structures. Inspection shows that $\mathrm{Mod}_{(\mathscr{S})}(S)$ and $\mathrm{Mod}_{(\mathscr{S})}(\mathrm{T}[S])$ are *identical* as subcategories of $\mathrm{Hom}(\|S\|, \mathrm{Set}) = \mathrm{Str}(\mathrm{L}_S)$; that is, in (14), the isomorphism is the identity.

The $\{\ulcorner\mathrm{Iso}\urcorner\}$-statements in $\mathscr{F}\mathrm{Sk}$ are the sketch-entailments of the form $\mathrm{Inv}[S,f]: S \to S'$, where S is any sketch, f is an arrow in S, and S' is obtained from S by adding a specification saying that "f is an isomorphism" (see Section 5). Let $\mathrm{Iso}[\mathrm{f}]$ denote the L_S-sentence $\forall b{:}B\exists! a{:}A.f(a) = b$; here, $f{:}A \to B$. Then, for any $M \in \mathrm{Mod}_{(\mathscr{S})}(S) = \mathrm{Mod}_{(\mathscr{S})}(\mathrm{T}[S])$, $\mathrm{Set}_{(\mathscr{S})} \models_M \mathrm{Inv}[S,f]$ iff $M \models \mathrm{Iso}[f]$. It immediately follows that

$$\mathrm{T}[S] \models_{\mathscr{S}} \mathrm{Iso}[f] \iff \mathrm{Set}_{(\mathscr{S})} \models \mathrm{Inv}[S,f]. \tag{15}$$

Let us write $S' \leq_f S$ for S' is a finite sub-sketch of S and f belongs to S'. When $S' \leq_f S$, $\mathrm{Inv}[S',f]$ is a finite approximation of $\mathrm{Inv}[S,f]$ and $\mathrm{T}[S'] \leq_{\mathrm{Iso}[f]} \mathrm{T}[S]$. $\mathrm{T}[S]$ is the directed union (colimit) of the theories $\mathrm{T}[S']$ for S' ranging over all S' such that $S' \leq_f S$.

Now assume that κ-compactness holds for the $\mathscr{S}$-semantics, and $\mathrm{Set}_{(\mathscr{S})} \models \mathrm{Inv}[S,f]$. Then, by (15), $\mathrm{T}[S] \models_{\mathscr{S}} \mathrm{Iso}[f]$, and there is some $S' \leq_f S$ for which $\mathrm{T}[S'] \models_{\mathscr{S}} \mathrm{Iso}[f]$, and by (15), $\mathrm{Set}_{(\mathscr{S})} \models \mathrm{Inv}[S',f]$, which shows that

(16) *κ-compactness for $\mathscr{S}$-semantics implies $\boldsymbol{S}_\kappa$-$\{\ulcorner\mathrm{Iso}\urcorner\}$-compactness for $\{\mathrm{Set}_{(\mathscr{S})}\}$ in* $\mathscr{F}\mathrm{Sk}$.

(15) immediately gives that

(17) *Abstract completeness for first-order logic with the $\mathscr{S}$-quantifier implies abstract completeness for $\boldsymbol{S} = \mathscr{F}\mathrm{Sk}$, $\mathscr{K} = \mathscr{K}[\{\ulcorner\mathrm{Iso}\urcorner\}]$, $\boldsymbol{P} = \boldsymbol{S}_{\aleph_0}$, and $\boldsymbol{C} = \{\mathrm{Set}_{(\mathscr{S})}\}$.*